

MU120 Unit 8



The Open University

Mathematics and Computing
A first level multidisciplinary course

Open Mathematics

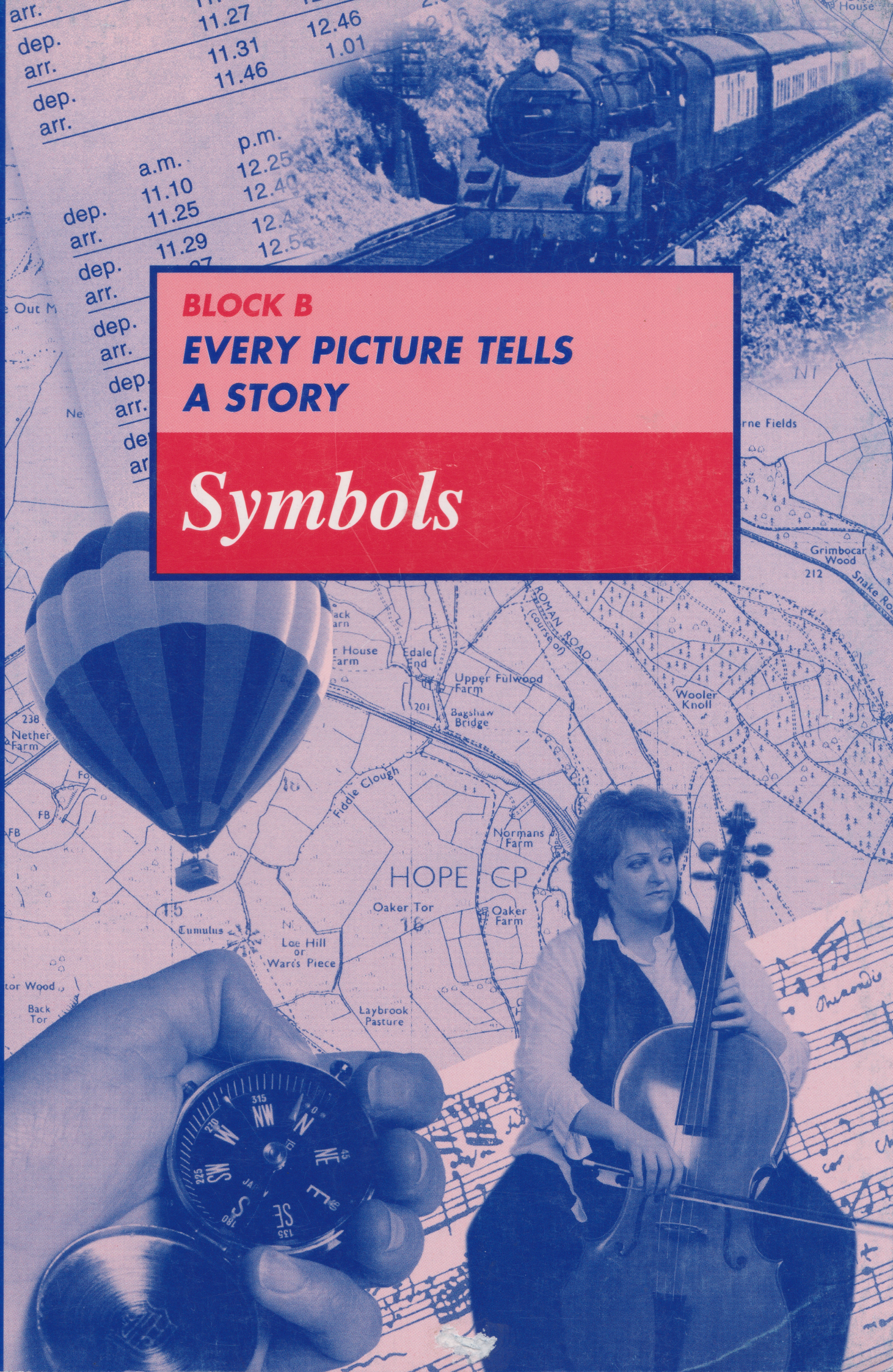
UNIT

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BLOCK B

EVERY PICTURE TELLS A STORY

Symbols





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Symbols

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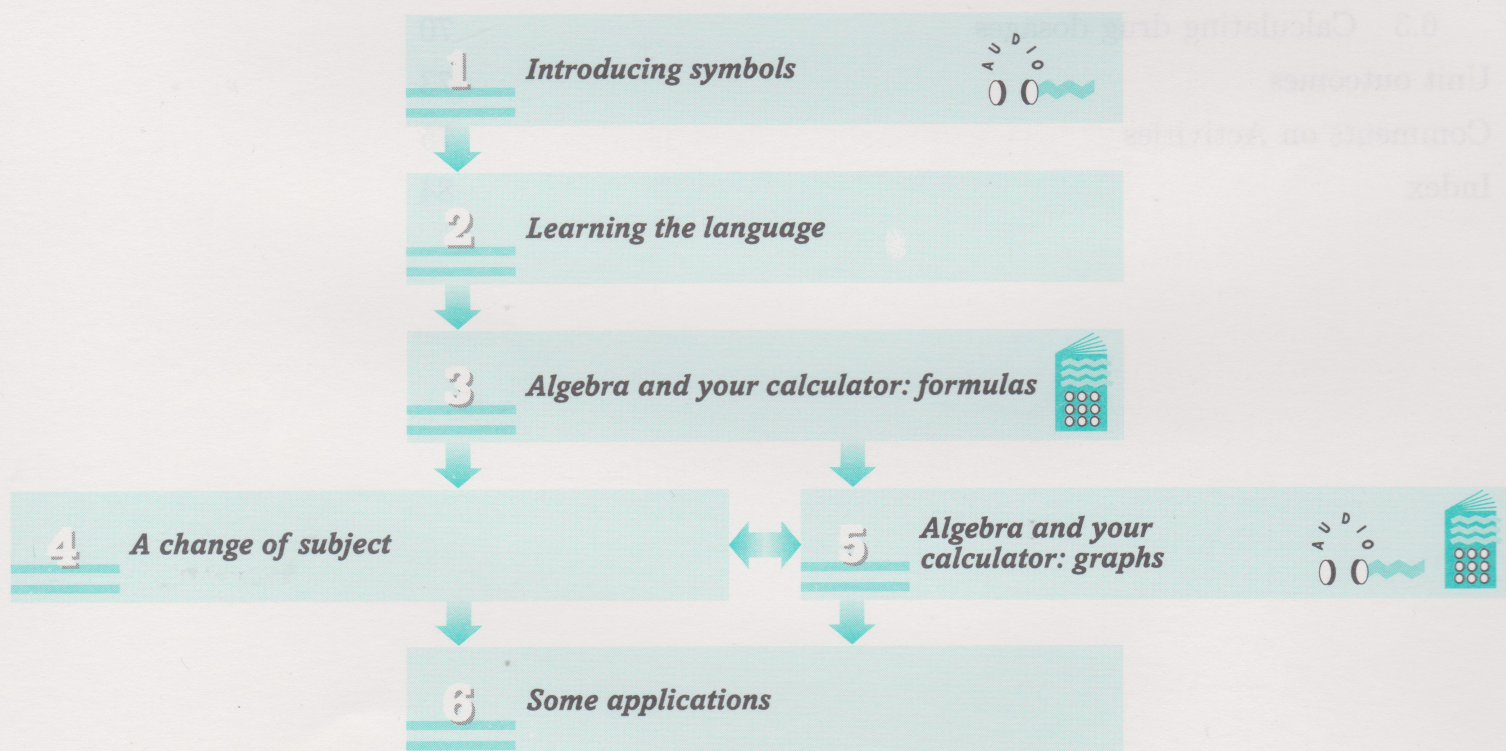
Study guide

Section 1 introduces the subject of the unit, which is the use of symbols in mathematics, and includes an audiotape band. In Section 2, you will discover more about the language of mathematical symbols. Section 3 involves the use of your calculator; you should study part of Chapter 8 of the *Calculator Book* here. Section 4 is concerned with manipulating equations.

You will need to use the calculator again in Section 5, which deals with graphs. There is more to be done in Chapter 8 of the *Calculator Book*, and there is some follow-up work in the unit. There is also a second band on the audiotape to listen to. Section 6 draws on all the work in the unit to investigate some problems of general interest.

Before you begin your work on this unit, you may want to take a few minutes to review your study schedule and your progress on this block. How are you progressing? Is your schedule realistic, or do you need to make changes for the rest of the block? Think about any modifications you may want to make to the way you study. How can you use your experience of studying so far to help implement those changes?

Use the planning and reviewing sheets to record any revisions or changes you intend to make as part of your 'action plan' for working on this unit. This sort of monitoring of your progress should be becoming second nature by now.



Summary of sections and other course components needed for *Unit 8*

Introduction

It is quite common in ordinary life to represent things by symbols, abbreviations, and other kinds of sign. Examples include: a Z-shape on a road sign, which warns you that you are approaching a double bend; abbreviations such as UK for United Kingdom and MP for Member of Parliament; car registration numbers; chemical formulas like H_2O ; the postcode MK7 6AA, which is part of the address of the OU or Open University; the shorthand used in knitting patterns, such as ‘work in garter st. in stripes thus: *6 rows in A and 2 rows in B* rep * to *’. Symbols are used on maps, as you saw in *Unit 6*: a P shows where there is parking, a small triangle where there is a triangulation pillar, a cross where there is a place of worship, and so on.

It is even more common to use symbols in mathematics. Using symbols, often letters, to stand for things is a very powerful mathematical technique. You have already come across the use of letters on a small scale in earlier units; when drawing graphs, for example, or when talking about the lengths of the sides of triangles. Some mathematical symbols, indeed, will be very familiar to you: $+$, $-$ and $=$ are obvious examples.

It is now time to explore in more depth the topic of using symbols to represent mathematical quantities and the way this opens up for working with them, called *manipulation*; this is the purpose of this unit. *Algebra* is the area of mathematics which concerns itself most with the specific manipulation of symbols representing quantities. You may wish to go back to listen again to the audiotope from *Preparing for Open Mathematics* ‘Communicating mathematically’, as well as having a quick look at the subsection on looking with symbols in *Unit 1*, before starting this unit.

The previous unit examined the use of graphical—pictorial and diagrammatical—ways of representing mathematical ideas, facts and relationships. This unit continues to develop the theme of representations; but it is concerned with what will seem, to begin with at least, to be a very different form of representation. However, symbolic and graphical representations are not as divorced from each other as they appear to be at first sight, and towards the end of the unit you will be back on more familiar territory when you will see how to move from one to the other.

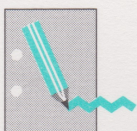
Despite using symbols in everyday life, many people often feel uncertain about using symbols in mathematics. Some of the activities associated with the Learning File invite you to think about and record your progress in learning and doing mathematics using symbols.

1 Introducing symbols



Aims This section aims to indicate some of the usefulness of symbols in mathematics, for expressing what happens ‘in general’. ◇

Perhaps when you saw the words ‘symbols’ and ‘algebra’ you immediately thought of complex strings of letters that meant very little to you and that you would not know what to do with. Or perhaps you have met symbols before in mathematical problems and never been particularly concerned by them. Many people, however, do find using symbols in mathematics difficult. As you work through this unit, you will learn about and gain experience in using symbols and indeed in designing formulas yourself which you can use to capture and explore different sorts of relationships. Activity 1 asks you to consider where you have used symbols before and gives you the opportunity to chart your progress on algebra through the unit.



Activity 1 Monitoring progress on symbols

Take a few minutes to think about your use of symbols in mathematics up to this point. Where have you used symbols in this course, and in other situations perhaps not related directly to mathematics? How would you assess yourself on algebra? What are your feelings about algebra? What criteria might you use? What evidence are you using to support your assessment and judgement?

Now ask yourself ‘What do I want to achieve by studying this unit and how will I know that I have?’ Again, try to be as specific as you can (the unit outcomes may help you here).

It may be helpful to keep your activity sheet with you as you work through the unit. On it, record progress you have made and the evidence you have to support this. If there are particular ideas or activities that you find difficult, make a note of them and at a convenient opportunity try to say why they are difficult to understand and record how you sorted them out (if you managed to!). (For example, you may have asked another student, found out at a tutorial session or worked it through by yourself.) Towards the end of the unit you can summarize your progress, drawing on your supporting evidence, on using symbols in mathematics.

1.1 Mind-reading

Listen to band 4 of Audiotape 2 while working through the following frames.



Frame 1

Think of a number Double it
 Add 5
 Double the result
 Subtract 2
 Divide by 4
 Take away the number you first thought of

Frame 2

Think of a number		7
Double it	2×7	14
Add 5	$14 + 5$	19
Double the result	2×19	38
Subtract 2	$38 - 2$	36
Divide by 4	$36 \div 4$	9
Take away the number you first thought of	$9 - 7$	2

Frame 3

Think of a number	
Add 4	
Take away the number you first thought of	

Frame 4

Think of a number	7
Add 4	$7 + 4$
Take away the number you first thought of	$(7 + 4) - 7$
And your answer is	4

Frame 5

Think of a number	7
Double it	2×7
Add 5	$2 \times 7 + 5$
Double the result	$2 \times (2 \times 7 + 5)$ or $2 \times (2 \times 7) + 2 \times 5$ or $4 \times 7 + 10$
Subtract 2	$(4 \times 7 + 10) - 2$ or $4 \times 7 + 8$
Divide by 4	$(4 \times 7 + 8) \div 4$ or $(4 \times 7) \div 4 + 8 \div 4$ or $7 + 2$
Take away the number you first thought of	$(7 + 2) - 7$ or 2

Frame 6

Add two numbers	$14 + 5 = 19$
Multiply by 2	$2 \times 19 = 38$

Multiply both numbers by 2	$2 \times 14 = 28$ $2 \times 5 = 10$
Add the results	$28 + 10 = 38$

NOTE

$$2 \times (14 + 5) = 2 \times 14 + 2 \times 5$$

$$2 \times (2 \times 7 + 5) = 2 \times (2 \times 7) + 2 \times 5 = 4 \times 7 + 10$$

Add two numbers	$28 + 8 = 36$
Divide by 4	$36 \div 4 = 9$

Divide both numbers by 4	$28 \div 4 = 7$ $8 \div 4 = 2$
Add the results	$7 + 2 = 9$

NOTE

$$(28 + 8) \div 4 = 28 \div 4 + 8 \div 4$$

$$(4 \times 7 + 8) \div 4 = (4 \times 7) \div 4 + 8 \div 4 = 7 + 2$$

Frame 7

Think of a number	
Double it	$2 \times$
Add 5	$2 \times + 5$
Double the result	$4 \times + 10$
Subtract 2	$4 \times + 8$
Divide by 4	$+ 2$
Take away the number you first thought of	2

Frame 8

Think of two numbers

Double the first

Add the second to the result

$$\begin{array}{l} 2 \times \square \\ 2 \times \square + \square \end{array}$$

Frame 9

Think of a number	Number
Double it	$2 \times \text{Number}$
Add 5	$2 \times \text{Number} + 5$
Double the result	$4 \times \text{Number} + 10$
Subtract 2	$4 \times \text{Number} + 8$
Divide by 4	$\text{Number} + 2$
Take away the number you first thought of	$(\text{Number} + 2) - \text{Number} = 2$

Frame 10

Think of a number	N
Double it	$2 \times N$
Add 5	$2 \times N + 5$
Double the result	$4 \times N + 10$
Subtract 2	$4 \times N + 8$
Divide by 4	$N + 2$
Take away the number you first thought of	$(N + 2) - N = 2$

And your answer is, indeed, 2 whatever the value of N .

Frame 11

Compare

$$2 \times (2 \times N + 5) = 2 \times (2 \times N) + 2 \times 5 = 4 \times N + 10$$

$$2 \times (2 \times 7 + 5) = 2 \times (2 \times 7) + 2 \times 5 = 4 \times 7 + 10$$

Compare

$$(4 \times N + 8) \div 4 = (4 \times N) \div 4 + 8 \div 4 = N + 2$$

$$(4 \times 7 + 8) \div 4 = (4 \times 7) \div 4 + 8 \div 4 = 7 + 2$$

1.2 Formulas

There would not be a great deal of point in studying algebra, if you only gained some understanding of how numerical ‘conjuring tricks’ which start off ‘Think of a number ...’ work. But algebra is far more powerful than this. Here is one example.

Unit 7 included the use of graphs for converting between different systems of units of measurement. Such a conversion graph provides one way of representing the relationship between measures of some quantity—a distance, for example, or a temperature—using different units. There is another way of representing a relationship of this kind, which is to use symbols to stand for the measurements involved. The relationship can then be expressed as a formula involving these symbols. The formula can be used to work out the conversion of measurements from one system of units to the other even more simply than the graph can.

Conversion formulas provide just one example of the very many ways in which symbolic formulas can be used to represent quantitative relationships. This section deals with some of the ways such formulas arise.

Conversion formulas

In *Unit 7*, you drew graphs to make conversions from one set of units to another: for example, from distances in miles to distances in kilometres, and from temperatures measured on the Fahrenheit scale to temperatures measured on the Celsius scale. Each of these conversion graphs has a corresponding formula, which can be used for making conversions without using the graph. One important use of symbols in mathematics is to write down such formulas in a compact, convenient and efficient way.

Suppose that you wanted a formula for converting distances given in miles into kilometres, perhaps for a French friend who did not feel happy with the peculiarities of the British imperial system. To obtain the formula, you would use the fact that 1 mile is 1.61 kilometres (to two decimal places). It can help to state first in words how the conversion is to be carried out: if you know a distance in miles, to find the corresponding distance in kilometres you multiply the number of miles by 1.61. To see how to represent the conversion symbolically, remember how the instructions for ‘Think of a number’ were treated. Following that lead, choose a letter to stand for the *number* of miles: M seems an obvious choice. The measure in kilometres (km) of a distance of M miles is $1.61 \times M$ km. It can be useful to have a symbol to stand for the kilometre measure as well as one for the mile measure: call it K km. Then the rule for converting from miles to kilometres can be expressed symbolically as follows: if a particular distance is M miles, and K km, then the numbers M and K are related by the formula:

$$K = 1.61 \times M$$

To find the equivalent in kilometres of a distance given (explicitly) in miles, using this formula, you just have to substitute the number of miles for M and do what the formula tells you to. For example, 5 miles is $1.61 \times 5 = 8.05$ km, and 27.5 miles is $1.61 \times 27.5 = 44.28$ km (to two decimal places).

Crossing the Channel from the UK to France or Belgium involves quite a lot of changing from one system of units to another, and generates a number of conversion formulas. Here is another example: 1 pound (lb) is 0.45 kilogram (kg), so if the mass of something is P lb and J kg, then the numbers P and J are related by the formula:

$$J = 0.45 \times P$$

It is better not to use K again, to avoid confusing kilograms with kilometres. (From the algebraic point of view, it is a nuisance that both of these metric measure words begin 'kilo'.)

Activity 2 Gallons and litres

One (imperial) gallon is 4.55 litres. Write down the formula relating the Imperial and metric measures of the same capacity.

Milk and beer are often sold in the UK in pints. One pint is an eighth of a gallon. What is this in litres?

Activity 3 From kilometres to miles

Many British people are unused to measuring distances in kilometres. Imagine you have some friends like this, who are planning to travel in mainland Europe, using a map on which all the distances are given in kilometres. Suppose they have asked you to help them convert these distances to miles.

To do so, you will need a formula like the one converting miles to kilometres, but the other way round. In other words, the required formula (for a distance of K km, and M miles, as before) will take the form:

$$M = ? \times K$$

What number goes before the $\times K$ in place of the question mark?

Use your formula to convert 40 km to the corresponding number of miles. Check whether your formula is correct by converting your answer back to kilometres using $K = 1.61 \times M$.

The use of formulas to convert measurements in different units is not restricted to conversions between metric and Imperial scales, of course. One interesting application is the conversion of measurements given in ancient documents or inscriptions into modern units. For example, the unit of measurement of length used by many ancient peoples, including Egyptians, Babylonians, Greeks, Romans and Hebrews, was the *cubit*. It was based on the length of the arm from the elbow to the extended fingertips, and is usually taken to be about 0.46 metres. However, different peoples seem to have used slightly different cubits: the Egyptian royal cubit, for example, was nearer 0.52 metres. This was subdivided into 28 *digits*, with 12 digits making a *span*.

For the standard cubit, take the formula relating the number of cubits C in a length to the number of metres L in the same length to be:

$$L = 0.46 \times C$$

Activity 4 How tall was Goliath?

According to the Bible (1611 translation): 'there went out a champion of the camp of the Philistines, named Goliath, of Gath, whose height was six cubits and a span' (I Samuel 17:4). Explain to a friend who does not know any algebra how to work out roughly how tall Goliath was in metres, assuming that the biblical cubit is the standard one.

Converting a temperature from $^{\circ}\text{Fahrenheit}$ to $^{\circ}\text{Centigrade}$ is more complex than the conversion formulas dealt with so far, because the zeros of the two temperature scales do not coincide. The two temperatures are not directly proportional to one another. In *Unit 7*, when you drew the graph for this conversion, you used two fixed points: the temperature at which water freezes, which is 32°F and 0°C , and the temperature at which water boils, which is 212°F and 100°C . The same data may be used to construct a formula for converting temperatures, as follows.

Between the freezing and boiling points of water there is a rise of 180°F , which is the same as a rise of 100°C .

So a rise of 180 degrees on the Fahrenheit scale corresponds to a rise of 100 degrees on the Celsius scale.

So each degree rise on the Fahrenheit scale corresponds to a rise of $\frac{100}{180}$ degrees on the Celsius scale; $\frac{100}{180}$ is the same as $\frac{5}{9}$.

In order to convert from Fahrenheit to Celsius, therefore, you need to do two things:

- ◇ compensate for the freezing point being 32°F as opposed to 0°C ; *and*
- ◇ take account of the fact that a rise of 1°F corresponds to a rise of only $\frac{5}{9}^{\circ}\text{C}$.

So the procedure for conversion is this: starting with the Fahrenheit temperature,

- (a) subtract 32 (to make the freezing points match); *and*
- (b) then multiply the result by $\frac{5}{9}$.

This set of instructions reads rather like part of a 'Think of a number' sequence, with the Fahrenheit temperature taking the role of the number you first thought of.

Activity 5 Temperature conversion

Work out the formula that gives the equivalent on the Celsius scale, $c^{\circ}\text{C}$, of a temperature of $f^{\circ}\text{F}$.

Check that the boiling point of water, 212°F , produces 100°C as its equivalent, according to your formula.

What is the Celsius equivalent of normal body temperature, which is often quoted in the UK as 98.4°F ?

Notice the different roles of the letters F and f , and C and c . F is an abbreviation of the name 'Fahrenheit' and merely labels the type of 'degrees' in use (similarly with the letter C from Celsius). There is nothing algebraic about this use of letters; it is merely an abbreviation for convenience and ease of memory. On the other hand, f and c are symbols for *numbers*, the *number* of degrees Fahrenheit (or Celsius). This use of lower-case letters here is the reverse of what was done (but for precisely the same reason) in the example of mile/kilometre conversions, because m and km are fixed abbreviations for particular units, so those particular letters are not available for use as algebraic variables in contexts involving distance. Hence, the capital letters M and K were used.

Activity 6 Temperature conversion: the other way

Convince yourself that the steps in converting from the Fahrenheit to the Celsius reading of a temperature are these: starting with the temperature on the Celsius scale, first multiply by $\frac{9}{5}$, then add 32 to the result.

What is the formula for the Fahrenheit equivalent of a temperature given on the Celsius scale?

Some second thoughts about which letters to use

Although it is convenient (and can help the memory) to use letters which have some affinity with what they stand for, it is not necessary to do so. The formula for converting from miles to kilometres may just as well be written as $A = 1.61 \times B$, or $y = 1.61 \times x$, as $K = 1.61 \times M$. Indeed, insisting on some standard way of choosing which letter to use will soon lead to difficulties, as with kilometres and kilograms. It is important to realise that it is the *form* which distinguishes the formula, not the particular letters that appear in it.

You have fifty-two letters of the (Roman) alphabet to choose from, counting capitals as different from lower case (small) letters, as is usually done. Some of these are reserved for particular uses, as you will find out later on. You can also call on the Greek alphabet if you know it—though nobody ever uses π for anything except the ratio of the circumference of a circle to its diameter, in the same way that although '6' is a symbol too, no one ever uses it for anything other than the number 'six'. That is more than enough in all but the most extreme circumstances; but even so, it is important to use symbols sensibly, which means that you must be careful to state clearly what each symbol means at the beginning of each new piece of work—and not to change what it stands for within it!

Formulas for travel

In *Unit 7*, you met two word formulas relating average speed to distance and time from the start of the journey. They were:

distance = speed multiplied by time

and

speed = distance divided by time.

Notice this is not just abbreviations of the words 'distance', 'speed' and 'time': d , s and t are numerical quantities now.

Using the shorthand d for the numerical distance (measured from the starting point in some units), t for time (measured from the starting time in some units), and s for average speed, the formulas can be written

$$d = s \times t$$

and

$$s = \frac{d}{t}$$

These formulas express the fundamental relationship between speed, distance and time, and many problems concerned with motion and travel can be solved with their aid. To use them for calculation in any particular instance, though, you would really need to know the value of one of these quantities. Suppose, for example, that you were planning a coach journey across France, mostly along autoroutes (motorways), and that you expected to average 95 km/h (kilometres per hour) on one particular morning.

- What is the formula relating the distance travelled to the time of travel in this case?

The speed is 95 km/h, so if the distance travelled is d km in a time of t hours, the formula will be:

$$d = 95 \times t$$

One complication that often arises, however, is that distance is not always measured directly from the starting point, or time from the starting time. Do you remember the saga of Alice and Bob from *Unit 7*? Well, they have two colleagues, Alastair, who works in Cardiff, and Bethan, who works in Edinburgh. Like Alice and Bob, Alastair and Bethan regularly make long car journeys, Alastair to Edinburgh and Bethan to Cardiff. They use the same route, and like to meet midway (on occasions when they are both travelling on the same day). Here are the relevant data for one such occasion. Alastair leaves Cardiff at 8.30 am and drives at an average speed of 45 mph. Bethan leaves Edinburgh at 10 am and drives at an average speed of 50 mph. The distance between Cardiff and Edinburgh is 400 miles (almost exactly).

Later in the unit you will be asked to work out at what time they meet. The task for now is to derive formulas for where each of them will be at a given time. Since in the end you will have to compare their times and distances, it is best to measure their distances from the same point and their times from the same time. Take Cardiff as the reference point for distances, and 10 am as the reference point for times. The first task is to find a formula for Alastair's distance from Cardiff, say a miles, at a time t hours after 10 am. At 10 am, he has already been travelling for $1\frac{1}{2}$ hours, at an average speed of 45 mph; he is therefore 67.5 miles from Cardiff. With each further hour that passes, he gets 45 miles further away from Cardiff. So his distance from Cardiff t hours after 10 am is given by the formula:

$$a = 67.5 + 45 \times t$$

(The expression ' t hours after 10 am' is potentially a bit misleading: it sounds as though the formula is restricted to whole numbers of hours, but of course this is not so. When using the formula for times that are not whole numbers of hours, however, you do have to remember to convert minutes to fractions of hours. To find how far Alastair is from Cardiff at 1.45 pm, for example, requires you to take t to be $3\frac{3}{4}$, or 3.75.)

What about Bethan? She is 400 miles from Cardiff initially; but she is travelling south, so she gets closer as time progresses. In fact, her distance, say b miles, from Cardiff t hours after 10 am is given by:

$$b = 400 - 50 \times t$$

Activity 7 How far from Edinburgh?

Who is closer to Edinburgh at 2.15 pm, Alastair or Bethan?

Activity 8 Average speeds

Suppose you set off in a car for a 300-mile journey as a passenger. After a little while you notice that the time is 10.00 (on the twenty-four hour clock), and the trip meter (which the driver can never remember to reset) registers 100 miles. You begin to wonder how you could estimate the average speed later on in the journey from subsequent readings of the mileage on the trip meter. You decide to devise a formula to work out the average speed, at t hours (on the twenty-four hour clock), when the trip meter registers d miles. What is the formula for your average speed, s mph, in terms of t and d ?

Suppose that at a quarter past eleven (in the morning) the trip meter reads 180 miles. What is your average speed since 10.00?

Finally, here are some exercises which link travel with the conversion of units.

Activity 9 Eurostar

- Devise a formula for converting speeds from kilometres per hour to miles per hour. (1 km is 0.62 miles to two significant figures.)
- Shortly after the Eurostar train leaves the Channel Tunnel *en route* for Brussels or Paris, it passes through Calais-Frethun station; and shortly after that, the conductor proudly announces over the intercom: 'Ladies and gentlemen, the driver has just informed me that the train is now travelling at its maximum speed of 300 kilometres per hour'. How much faster is that than the current (1995) train speed record in Britain, which is 154 mph?

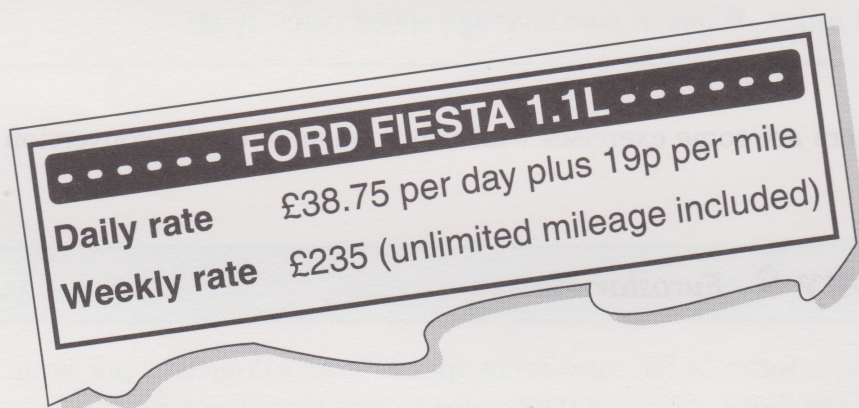
- (c) For scientific purposes, speeds are usually measured in metres per second. Devise a formula for converting from mph to metres per second. What is a speed of 60 mph in metres per second?
- (d) As the Eurostar train speeds through the Pas de Calais, the miles flash quickly past. Find a formula which will enable you to work out how many seconds it takes to cover a mile, if you are given the speed in km/h, and use it to find out:
- (i) how long it takes Eurostar to cover a mile;
 - (ii) how long it takes the Earth to cover a mile in its orbit through space (the Earth's orbital speed is about 107 000 km/h);
 - (iii) how long it takes light to travel a mile (the speed of light is about 1 100 000 000 km/h).

Other formulas

The usefulness of formulas is not restricted to converting from Fahrenheit to Celsius and to working out average speeds. To give you a taste of some other possibilities, here are a couple of activities devising formulas to describe some quite different situations. Your work on these activities should help you consolidate what you have just learned about the use of formulas and turn it to advantage in devising formulas to describe new situations.

Activity 10 Car hire

Judy needs to hire a car from Monday to Friday inclusive, and sees the following advertisement in her local paper.



- Which is the better deal?

It probably depends on how many miles she drives. To help her work out which would be the better deal, Judy could use a formula to show how the cost of hiring the car at the daily rate depends on how many miles she will drive.

- (a) What would be the cost of hiring the car for five days at the daily rate, without taking the mileage into consideration?

- (b) What would be the cost, $\pounds p$, of hiring the car for that period, at the daily rate, if she drives m miles?
- (c) Suppose that Judy expects to drive about 150 miles in all during the five days. Would she be better off with the daily or the weekly rate? She may have to make an additional trip of 100 miles during the week. If she were to travel 250 miles, which would be the cheaper option then?

Activity 11 Hooper's rule

Roadside and field-boundary hedges in England can be very ancient. There is an interesting way of estimating the age of a hedge which was devised by Dr Max Hooper in 1974. Dr Hooper counted the number of different species of tree and shrub in 227 established hedges whose ages he knew from written records. He found that the more different species there are, the older the hedge is likely to be.

Moreover, he came up with a formula which best fits the data that he collected, and which has been used successfully since to date other hedges known to be old.. The formula, which is known as Hooper's rule, works like this: you take the average number of species in 30 yards of the hedge, multiply it by 110, and add 30 to the result. This gives the approximate age of the hedge in years.



- (a) Express this formula symbolically.

To find the age of a hedge using Hooper's rule, you mark it off into consecutive 30-yard sections, count the number of different species of tree or shrub in each section, find the average (mean) value, and apply the formula.

- (b) I carried this out in 1995 for a hedge along the road from Aspley Guise to Husborne Crawley, two villages in Bedfordshire. My results are summarized in the following table.

Species of tree or shrub	Section							
	1	2	3	4	5	6	7	8
Ash	•							
Elder					•	•	•	•
English elm						•	•	•
Hawthorn	•	•	•	•	•	•	•	•
Maple							•	
Oak		•		•				
Wild plum	•		•	•			•	•
Wild rose	•		•	•				
Wych elm							•	

The presence of the symbol • in the table means that that species is present in that stretch of the hedge.

When does the hedge date from, according to Hooper's rule?

Afterword on algebra

Mathematics uses letters as symbols to stand for variable quantities and indeed for all sorts of other things; it employs various methods of calculating with symbols, as a substitute for working directly on the things themselves that the symbols represent.

Algebra as an explicit set of techniques for working with mathematical symbols has only been available in the West since the sixteenth century. But its use has become widespread and has gained ascendancy over more visual, geometric methods.

Algebra has frequently been seen as a stumbling block for people of all ages trying to learn mathematics, a prime example of manipulating symbols for no apparent reason.

Stand firm in your refusal to remain conscious during algebra. In real life, I assure you, there is no such thing as algebra.

(Lebowitz, F. (1982) *Social Studies*, London, Arrow, p. 27)

Comic Fran Lebowitz offered this as one of her 'tips for teenagers'. The French expression equivalent to 'It's all Greek to me', meaning something is incomprehensible, translates to 'It's algebra for me'.

Acknowledging this is not intended to put you off, but to encourage you to work at becoming more fluent at an activity which lies near the centre of mathematics—working with symbolic expressions. Transformation of

symbolic expressions is a key power of algebra. As author Thomas Carlyle put it:

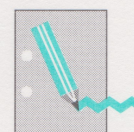
In a symbol there is concealment and yet revelation.

(Carlyle, T. (1836) *Sartor Resartus*, Oxford University Press, p. 166)

Fluent users of algebra report two awarenesses when working with algebraic expressions: being able to see them as structured strings of symbols *and* seeing them as descriptions connected with some ‘reality’ or situation they are concerned with. Maintaining this dual perspective is of central concern when working on mathematical symbols at whatever level, and can place a heavy burden on novices. Algebra offers both a language and a means of calculation.

Activity 12 Pros and cons of algebra

Start to draw up a list of the advantages and drawbacks of algebra, using the printed response sheet which is divided into two columns, one headed ‘Pros of algebra’ and the other ‘Cons of algebra’. List as many advantages (pros) of algebra (to you personally or to the world in general) as you can think of in the first column, and as many of its disadvantages as you can think of in the second. You can put specific examples of uses of algebra that appeal to you in the pros column, and specific examples of things that have discouraged you in the cons column. File the sheet in your Learning File. You should add to the sheet throughout this unit—indeed, throughout the course.



This section began with an introduction to the basic idea of algebra—using a letter to stand for an unknown number—via a kind of ‘conjuring trick’ in which someone is asked to ‘think of a number’. The result of carrying out each step can be represented by a formula involving N , the number first thought of. These ‘tricks’ work because the final formula always reduces to a particular number independent of N . By using an algebraic formulation and then simplifying and rearranging it, you can see, more easily than by any other method, *why* the result will be the same irrespective of which number is first thought of. The same fundamental idea, of converting a description of a sequence of calculations from words to symbols, underlies the use of algebra to formulate relationships between measurements in different units, between distances and times in travel problems, and in many other situations.

If you reflect on what you have been doing in this section, both with ‘Think of a number’ tricks and with formulas, you will see that it has consisted basically of taking a sequence of arithmetic operations described in words, and converting it into symbolic form by using letters to represent the variable quantities involved and other symbols for the arithmetic operations involved. In other words, this initial exposure to algebra has involved a process of translation, from the English language into a new symbolic language, the language of algebra.

Outcomes

After studying this section, you should be able to:

- ◇ represent the result of each step of a ‘Think of a number’ sequence by a formula involving a letter, such as N , to stand for the initial number;
- ◇ convert relationships (for example, between quantities such as measurements in different units) specified in words, into symbolic form, making your own choice of letters to stand for the quantities involved (Activities 2 to 11);
- ◇ recognize the benefits and drawbacks of the use of symbols in mathematics (Activity 12);
- ◇ provide examples of work demonstrating self-assessment.

2 Learning the language

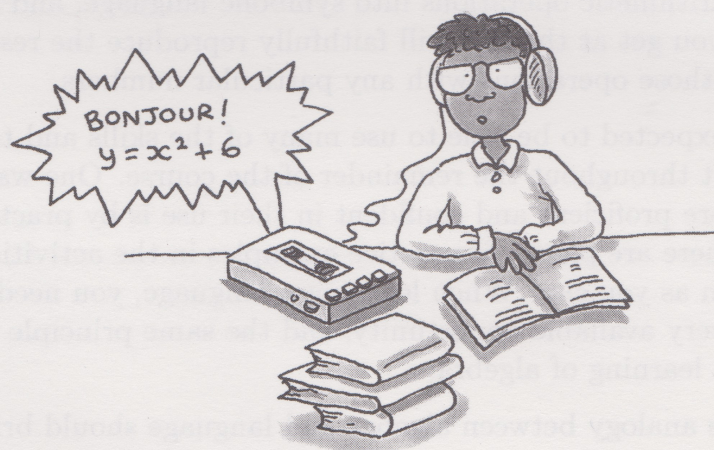
Aims The aim of this section is to enable you to say with greater conviction: ‘Mathematics—I speak it fluently’. ◇

The language of mathematics

To talk of *translating* arithmetic operations into symbolic form implies that you could think of algebra as a language; and in a sense it is a language. It is a language for expressing mathematical ideas, generalizations and relationships. In order to study and do mathematics successfully, you have to learn to speak this language; and the better you can speak it, the easier you will find the study of mathematics. To take just one example: learning the language of algebra is necessary to get the most out of your calculator, as this is the language it understands and responds to.

A language comprises its words, its forms, its uses and its meanings. The forms are the ways words can be put together to create meaningful phrases and sentences. Grammar is an attempt to express the regularities of these forms explicitly. Change the form and you frequently change the meaning. For instance, many languages including English are sensitive to word order: ‘the cat sat on the mat’ means something different from ‘the mat sat on the cat’, even though it uses exactly the same set of words. You can use language to interact with the world. For living languages, these uses involve *promising*, *swearing*, *objecting*, *marrying*, and many many more. Mathematical uses include expressing a conjecture, formulating a proof, exposing a faulty line of argument, and manipulating an expression.

Like all languages, algebra has its own vocabulary and rules of grammar. This section is devoted to introducing to you some of this vocabulary and grammar explicitly, so that you can develop greater fluency in reading and writing algebra.



The ‘words’ of algebra are the individual symbols—the letters which stand for numerical quantities, and the signs which represent mathematical operations like addition, multiplication or raising to a power. These words can be strung together to form phrases and sentences, just as with the words of any language. But words cannot be combined meaningfully in any old way: there are conventions that have to be respected if the result is to make sense.

The role of the rules of grammar of a language is to ensure that communication is clear and meaningful. Grammatical phrases—those formed according to the rules of grammar—have a better chance of being understood than ungrammatical ones. Grammar is even more important in algebra than it is in ordinary language: an ungrammatical phrase in algebra usually makes no sense at all.

Symbols are used to stand for things in many different situations, not just in algebra. A symbol often stands for a specific, particular thing. In algebra, on the other hand, a letter symbol usually represents an unspecified, general thing—some number, it does not matter which. Algebra is a language which is particularly well adapted to making general statements, and to making them precisely and concisely.

By letting N stand for ‘the number you first thought of’, you can summarize *all* possible results of carrying out the sequence of calculations in a ‘Think of a number’ task. By letting f stand for an unspecified Fahrenheit temperature, you can write down a formula for converting *any* temperature at all from Fahrenheit to Celsius. It is this type of concise generalization that makes algebra such a powerful tool. And the conciseness allows an ease of manipulation: changing the form of an expression while preserving the meaning. The new algebraic form usually makes some property or relationship easier to see.

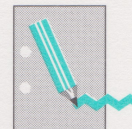
Another important point about algebra is that you can represent operations of arithmetic—addition, subtraction, multiplication and division, finding squares, square roots, and so on—in symbolic terms. These operations may not look quite the same when you write them algebraically as when you work with them with specific numbers, but they function in essentially the same way. This means that you can translate a sequence of arithmetic operations into symbolic language, and know that the formula you get at the end will faithfully reproduce the results of carrying out those operations with any particular numbers.

You will be expected to be able to use many of the skills and techniques from this unit throughout the remainder of the course. One way of becoming more proficient and confident in their use is by practising lots of examples. There are plenty of practice examples in the activities: do as many of them as you can. When learning a language, you need to practise using it at every available opportunity, and the same principle applies to the confident learning of algebra.

Stressing this analogy between algebra and language should bring certain similarities to light. If you can train yourself to see the grammatical

structure of formulas, to treat algebra as though it were like some familiar foreign language, and to translate readily back and forth between your native tongue and symbolic ways of expressing relationships, then you should be able to speak mathematics more fluently.

Activity 13 *Techniques of algebra*



Learning to use algebra, like learning a language, involves learning to express meanings through structures. The most important of these structures are italicized in the text and included in the Index at the end of the unit. To help you learn and remember them (and any other terms that you think you need help to remember), make a note of them and how to use them on the Handbook activity sheet. You can, of course, add to this and change it at any time, and it will provide a growing and steadily improving reference to assist your study.

Expressions

A collection of symbols like $4 \times N + 10$, $1.61 \times M$ or $\frac{9}{5} \times c + 32$ is called an *algebraic expression*, or simply an expression. An algebraic expression contains one or more symbols standing for unspecified numbers: these quantities are often called *variables*. An expression is the algebraic equivalent of an English phrase.

An algebraic expression can be thought of as a way of codifying a calculation, or the end result of a sequence of calculations. To see it in this guise, imagine replacing each symbol, wherever it appears in the expression, by a particular number, in order to carry out the requisite arithmetic operations for that particular case. For example, the expression $4 \times N + 10$ codifies the operations ‘take your number (whatever it is), multiply it by 4, and add 10 to the result’. So if you replace N by 0.5 in the expression $4 \times N + 10$ you will obtain $4 \times 0.5 + 10$ (which, of course, is 12).

Replacing a variable by a number like this is called *substitution*; calculating the number obtained by substitution in an expression is called *evaluating the expression*. So in the above case 0.5 was substituted for N in the expression $4 \times N + 10$, which was evaluated to give 12.

The number that you substitute for a variable is often called the *value of the variable*, and the number that you get as a result of evaluating an expression is often called the *value of the expression*. So another way of expressing the above example would be:

The variable N was given the value 0.5; the expression $4 \times N + 10$ was then evaluated, and found to have the value 12.

Sometimes the term ‘numerical value’ is used instead of ‘value’, to emphasize that the value is a number.

Finally, the equals sign is often used to indicate the assignment of a value to a variable, and the subsequent result of evaluating an expression. You could say *either*

the value of $4 \times N + 10$, when N takes the value 0.5, is 12;

or

when $N = 0.5$, $4 \times N + 10 = 12$.

Activity 14 Evaluation

- What is the value of $2 \times p - 2$ when p takes the values
(i) 3; (ii) 7.165; (iii) -1 ?
- What is the value of $3 \times M + 5$ when $M = 3$?
- What is the value of $2 + b - 2 \times b$ when b is 5?
- Complete the sentence: when $x = 6$, $1 + \frac{1}{3} \times x = \dots$

Does it make sense?

A collection of symbols like $4 \times N + 10$ is called an algebraic expression. But some collections of symbols make no sense: for example, $4 \times N +$, $2 \times -p 2$, or $2 \times M + \times N$. Meaningless collections of symbols do not deserve to be called algebraic expressions, and are not—just as ‘meaningless of collections words’ would not be called an English phrase. You will have to deal with more complicated expressions as the course progresses; it is therefore important to learn to scan collections of symbols to check that they make sense, that they are correctly structured, which is quite independent of whether or not they are *true* mathematical statements. The question to ask yourself is this: if I were to substitute a particular numerical value for the variable (or particular values for the variables if there are several) in this collection of symbols, would I be able to calculate a numerical result which would count as the value of an expression? If you try substituting $N = 0.5$ in $4 \times N +$, you soon see that you do not get a numerical value for the result! Another check is that each operation $\times$, $\div$, $+$ or $-$ must have a number after it to multiply by, divide by, add or subtract. An expression can start with a minus sign though; in such a case, the sign is interpreted just the same as when it signifies a negative number. Some people like to read the expression aloud when checking to see if it makes sense or if there are some words missing.

Activity 15 Sense or non-sense

Which (if any) of the following collections of symbols do not ‘make sense’?

- $a + 2 -$
- $6 \times m + 35 - m$
- $+4 \times N + 10$
- $-4 \times N + 10$

Doing without the multiplication sign

In arithmetic, multiplication is usually indicated with the sign $\times$; the same system has been used so far in algebraic expressions. But it is not always necessary to use $\times$ to show that multiplication is required: for example, $5(3 + 2)$ means $5 \times (3 + 2)$. It is normal practice to omit multiplication signs in algebra when they are not necessary; and it turns out that this is the case most of the time. So, for example, write $4N$ instead of $4 \times N$ for 4 multiplied by N . This convention applies when a number multiplies a symbol, but not when a number multiplies another number: 24 continues to mean twenty-four, never 2 times 4. This last observation shows why you have to use the multiplication sign in arithmetic: because the decimal system for representing numbers makes use of the relative positions of the digits. This place system does not operate in algebraic expressions, so the multiplication sign is unnecessary. The expression $2ab$ does not ever mean 'two hundred, a tens, and b units', but always $2 \times a \times b$.

Note that *it is not wrong* to use the multiplication sign in algebra; it is just convention that the sign is usually omitted, to save writing. Indeed, there are occasions when a multiplication sign *is* used, for emphasis, or to avoid ambiguity. If you find that putting in multiplication signs helps you to understand algebraic expressions, by all means do so. Multiplication is sometimes represented in algebraic expressions by a dot, thus: $4 \cdot N$. This usage is worth noting in case you come across it somewhere else; and it provides another option for you in your own work.

In an algebraic expression, $N4$ would mean the same as $4N$, namely $N \times 4$ or $4 \times N$. But it is usual to write the number first and the symbol second, so $4N$ is preferable to $N4$. This is not a grammatical rule, it is more a point of style, like not ending your sentences with 'up' or 'with'. (A convention which Winston Churchill famously rejected by saying sarcastically that it was something 'up with which I will not put'.)

The terminology and notation for repeated products is carried over to algebra without change. Thus, the product of (say) x with itself is written x^2 , rather than xx , and called ' x squared'. (However, historically, xx and xxx were commonly used notations.) Similarly, p^3 (p cubed) represents $p \times p \times p$, and $7R^2$ is $7 \times R \times R$. Expressions like x^2 , x^3 , and x^4 are called *powers of x* .

Root signs are used in algebra just as they are in arithmetic: $\sqrt{N}$ is the (positive) square root of the number N (N must be assumed to be positive for this to make sense); $\sqrt{5x^2}$ is the square root of $5x^2$; $\sqrt[3]{k}$ is the cube root of k ; and $\sqrt[3]{p^2}$ is the cube root of the square of p .

One situation where you may have to use a multiplication sign is if you want to multiply a number by an expression, and the expression starts off with a number: for example, if you want to multiply 2 by $4N$. Omitting the multiplication sign will not do here: $24N$ means twenty-four times N , not 2 times $4N$. To avoid ambiguity here, write $2 \times 4N$ or $8N$. This holds because $2 \times 4N$ is the product of three numbers, $2 \times 4 \times N$, and it is a basic property of multiplication that in a situation like this it does not matter which product you evaluate first.

Activity 16 *Down with multiplication*

Express the following without $\times$ signs.

- (a) $3 \times N$ (b) $4 \times \frac{1}{6} \times z$
 (c) $8P \times 7P$ (d) $24 \times 2x$.

Simplifying expressions

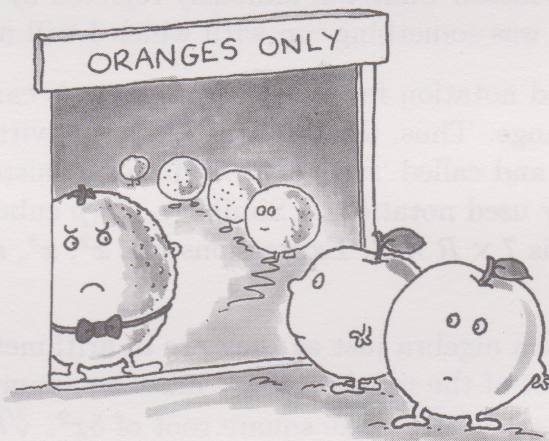
Some expressions are more complicated than they need to be: for example, $2 \times 4N$ may be more simply written as $8N$. Another kind of expression which can be simplified in an analogous way is one which is the sum of two multiples of the same variable, for example $2x + 5x$. Since 2 lots of something plus 5 lots of *the same thing* come to 7 lots of it in total, this expression can be written more simply as $7x$.

Try a few simplifications like these for yourself.

Activity 17 *Simplification*

Simplify the following expressions as far as possible.

- (a) $12 \times 2A$ (b) $6B + 7B + 8B$ (c) $4x + 2 \times 3x$
 (d) $11y - 5y$ (e) $2p^2 + 5p^2$ (f) $6A + 7A^2$



The last of those problems illustrated one general principle of simplification, the 'apples and oranges' principle: you cannot add apples to oranges. So you have to be careful, when simplifying an expression which contains different powers of the variable, not to mix the separate powers up. However, it is none the less often possible to do some simplifying of an expression which contains several powers.

This principle may be expressed more carefully as follows. A mini-expression which is just a multiple of some variable, or a power of it, is called a *term*: $6A$, $7B$ and $8B^3$ are all examples of terms. It makes the explanation simpler if the variable itself is considered as a power, namely the first power; x can be thought of as x^1 . Terms which are multiples of the same power of the variable are called *like terms*: $7B$ and $8B$ are like terms, but $7B$ and $8B^3$ are not. Then, when you are simplifying an expression which involves different powers, you must make sure to combine like terms only.

The numerical factor in front of a term is often called the *coefficient*; the coefficient of A in the expression $6A + 7A^2$ is 6. Note the rule for adding like terms is to add the coefficients.

There is one slight oddity about this: a 'bare' term, like x or Z^2 , does not appear to have a coefficient. But in fact it does: its coefficient is simply the number 1. There is rarely any good reason for writing $1x$, since $1 \times x$ is just x . But it is sometimes helpful to remember, when you see an expression like x , that you could think of it as $1x$ if you wanted to.

Activity 18 Oversimplification

Simplify the following expressions as far as possible.

- (a) $x + 8x + 3x^2 - 2x^2$
- (b) $x + 8x^2 + 3x - 2x^2$
- (c) $1 + 2a + 3a^2 + 4a + 5$
- (d) $2 \times 3A + 2A^3 + 2^3A$

Brackets

Suppose that you wanted to produce a symbolic version of the following calculation:

take a number, add 10 to it, then multiply the result by 4.

If N is the number, then adding 10 gives $N + 10$; and you might be tempted to write just $4N + 10$ for what you get by multiplying the result by 4. But this would be wrong: $4N + 10$ represents something quite different. If you need to be convinced that this is so, just consider what the instructions 'take a number, add 10 to it, then multiply the result by 4' lead to when the number you take is 0.5: you get 4×10.5 , which is 42; but the value of $4N + 10$ when $N = 0.5$ is certainly not 42. Nor is $N + 10 \times 4$ any good: this does not give 42 when $N = 0.5$ either.

Activity 19 Re-evaluation

Evaluate $4N + 10$ and $N + 10 \times 4$ when $N = 0.5$.

The reason neither of these expressions corresponds to ‘add 10 to N , then multiply the result by 4’ is that, in the case of $4N + 10$, only the N is multiplied by the 4, while, in the case of $N + 10 \times 4$, only the 10 is so multiplied. What is needed is a way of indicating that it is the *result* of adding 10 to N that gets multiplied by 4.

The best way of doing this is to use brackets: $4 \times (N + 10)$ and $(N + 10) \times 4$ both mean *first* add 10 to N , *then* multiply the answer by 4. The brackets signify that what they contain is to be treated as a whole.

The convention about omitting the multiplication sign also applies when one of the factors is an expression in brackets, because there is no possible ambiguity: thus $4(N + 10)$ means the same as $4 \times (N + 10)$. However, it is usual to put the number in front of the bracket, as before: $4(N + 10)$ is preferable to $(N + 10)4$. So the usual way of writing the expression which means ‘first add 10 to N , and then multiply the result by 4’ is $4(N + 10)$.

Always remember that if you have an expression to evaluate which contains brackets, evaluate what is inside the brackets completely before taking account of what lies outside them.

Activity 20 Brackets

- (a) Evaluate $4(N + 10)$, $4N + 10$ and $N + 10 \times 4$ when $N = 7$.
- (b) In the ‘Think of a number’ sequence on page 7, you had to double $2N + 5$ at one stage, and divide $4N + 8$ by 4 at another. Express each of these steps with brackets.

The use of brackets provides another way of dealing with the problem of how you represent the result of multiplying 2 by $4N$: you can write it as $2(4N)$. When you are multiplying several numbers (or symbols, which represent numbers) together, it does not matter in which order you do the multiplication. This fact has been used when it was pointed out earlier that $2 \times (4 \times N)$ (or $2(4N)$) is the same as $(2 \times 4) \times N$ (or $8N$).

An expression enclosed in brackets can be treated as a single entity. So you can treat $6(N + 10)$ as a term; and $2(N + 10)$ is then a like term. The expression $6(N + 10) - 2(N + 10)$ may be simplified to $4(N + 10)$.

Activity 21 Simplification with brackets

(a) Express the following expressions without multiplication signs.

(i) $(m + 3) \times 9$ (ii) $(m + 3) \times (9 - 2 \times m)$

(b) Simplify the following expressions.

(i) $2(m + 3) + 3(m + 3)$ (ii) $5(4N + 10) - 2(4N + 10)$

(iii) $3(9 - 2 \times z) + (9 - 2z) \times 2$

One of the most important functions of brackets is to eliminate ambiguity. It is essential that mathematical formulas are clear and unambiguous; it is therefore important for you to recognize the usefulness of brackets in constructing algebraic expressions. There is a golden rule about the use of brackets: it is better to put them in than leave them out. Brackets are cheap: use them often. When you have to write down an algebraic expression which represents some sequence of operations, there might be some doubt about how the expression could be interpreted, so see if you can use of brackets to prevent a misunderstanding.

A brief note on equality

You saw that $6(N + 10) - 2(N + 10)$ may be simplified to $4(N + 10)$. It may have struck you that this is rather a long-winded way of putting things; what it means, after all, is that the two expressions are to all intents and purposes the same, so why not write:

$$6(N + 10) - 2(N + 10) = 4(N + 10)?$$

This is indeed the usual practice, adopted from now on. Just note, however, that the equals sign here has a different meaning from what it has when you say 'when $N = 0.5$, $4(N + 10) = 42$ '. This is a claim about the value of an expression when its variable is assigned a particular value. Writing $6(N + 10) - 2(N + 10) = 4(N + 10)$, on the other hand, is to claim that the two expressions are just different ways of writing the same thing; another way of putting this is that $6(N + 10) - 2(N + 10)$ and $4(N + 10)$ always take the same value, whatever the value of N .

Expanding brackets

It is very common for formulas to involve brackets. Sometimes it is more convenient to have the formula with brackets and sometimes it is not; so it is useful to be able to rewrite a formula containing brackets without them. This is called *multiplying out the brackets* or *expanding* them. You may remember from the 'Think of a number' sequence from the audiotape that multiplying out the brackets in the expression $2(2N + 5)$ gives $4N + 10$. Note carefully that when the brackets are expanded, the 2 outside the brackets multiplies both of the terms inside them. Similarly, when

multiplying out the brackets in the expression $3(4X - 3)$, the factor 3 must multiply both of the terms inside the brackets; so the expression becomes $3 \times 4X - 3 \times 3$, which is to say that:

$$3(4X - 3) = 12X - 9$$

Activity 22 *Doing without brackets*

Express the following without brackets.

(a) $2(m + 3)$ (b) $6(2X + 1)$ (c) $5(9 - 2z)$

(d) $n(n + 2)$ (e) $6X(2X + 1)$

Example 1 *Removing brackets*

Express $(n + 2)(n + 3)$ without brackets.

Where to start? Well, remember that $(n + 2)$ is just some number, though a number written in a complicated way; and you know how to multiply brackets out by a number. So you can use the expansion of (say) $4(n + 3)$ as a pattern for working out $(n + 2)(n + 3)$. Now:

$$4(n + 3) = 4n + 4 \times 3$$

so

$$(n + 2)(n + 3) = (n + 2)n + (n + 2) \times 3 = n(n + 2) + 3(n + 2)$$

You are now halfway to a solution. It remains to expand the two brackets in this expression to eliminate the brackets altogether.

$$n(n + 2) + 3(n + 2) = n^2 + 2n + 3n + 6 = n^2 + 5n + 6$$

And that is the required answer. It cannot be simplified further, since none of the terms are like any of the others.

The key to this solution is that an expression like $(n + 2)$ can be treated as a single entity, as well as a compound expression made up of two terms, n and 2. Admittedly, you have to be somewhat two-faced in the way you approach the calculation. The two expressions $(n + 2)$ and $(n + 3)$ are not treated on an even footing: to begin with, $(n + 2)$ is treated as a single entity while $(n + 3)$ is treated as a compound expression. However, this does not invalidate the conclusion. But it might raise questions in your mind: what would happen if I did things the other way round? Suppose I started by treating $(n + 3)$ as the single entity and $(n + 2)$ as the compound expression—what then? Would I get a different answer?

Activity 23 Multiplying out brackets

Multiply out the brackets in the expression $(n + 2)(n + 3)$ the other way round, treating it like

$$(n + 2) \times 5 = n \times 5 + 2 \times 5,$$

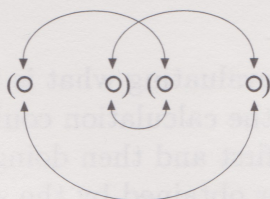
and check that you get the same result as before.

- How did you react to that demonstration? Was it a waste of time? Did you think it was necessary? Did you think it was useful?

There are people who worry about things like this—who ask questions like ‘How do you *know* that you’ll get the same answer whichever way you do it?’ They are called pure mathematicians, and, among other things, they do for mathematics what theoretical physicists do for physics, theologians do for religion, and philosophers do for knowledge of all kinds: they try to ensure that the foundations of the subject are sound.

When you think about it, there are many different ways of writing down the same algebraic expression: for example, $4(n + 3)$, $4n + 12$, $4(3 + n)$, $(3 + n) \times 4$, $12 + 4n$ are all forms of the same expression. These are not necessarily all the possible forms of this expression: there may be others. You can check that they are just different ways of writing the same thing by observing that each can be transformed into any of the others by carrying out some valid algebraic operations, such as multiplying out brackets, or reversing the order of terms in an addition or a multiplication.

In practice, whenever you multiply out brackets, make sure that everything in one bracket is multiplied by everything in the other, like this ‘face’.



Activity 24 More multiplying out

Express the following without brackets.

- $(y + 7)(y + 1)$
- $(p + 2)(p + 2)$
- $(p + 2)^2$
- $(2x + 7)(x + 1)$
- $(4p + 2)(3p + 2)$

Activity 25 *Odd one out*

Find the odd one out in each of the following.

$$\begin{array}{ll} \text{(a)} & (m+3)(m+5) \qquad 5(m+3) + m^2 + 3m \\ & (5+m)(3+m) \qquad m^2 + 8 + 15m \end{array}$$

$$\begin{array}{ll} \text{(b)} & 2 + x + 3x^2 + 6x \qquad (3x+1)(x+2) \\ & 3x(x+2) + 2(3x+1) \qquad 3x^2 + 7x + 2 \end{array}$$

Activity 26 *How many ways?*

How many ways, different in form but equivalent in meaning, can you find of writing the expression $x^2 + 2(x+3) + 3x$, in five minutes?

You may want to pause here to check that you have included on your Handbook activity sheet everything that you have met so far that you think could be useful and important to you for your further study.

Brackets and negative numbers

The claim that a negative number times a negative number is a positive number, or 'a minus times a minus is a plus', is (on the face of it) one of the mysteries of mathematics. There are many ways of illustrating why this must be so: being let off, or subtracting, a debt is equivalent to being given money, is one favourite. There is another, more serious, explanation, which some people find convincing: multiplying out brackets does not work consistently unless a negative number times a negative number is positive.

Consider, for example, the following calculation:

$$16 - 4(3 - 1)$$

If this calculation is done by evaluating what is in the brackets first, the result is $16 - 4 \times 2 = 8$. But the calculation could also be done by multiplying out the brackets first and then doing the necessary addition and subtraction. If the answer obtained by the second method is different, then something is badly wrong: mathematics is not going to be much use if it is not consistent!

So consider what happens when the brackets are multiplied out. It is necessary to work out $-4(3 - 1)$. Now -4×3 must be $-(4 \times 3)$, which is -12 .

► But what is $-4 \times (-1)$?

Well, there are two possible options, either $+4$ or -4 . If $-4 \times (-1)$ were -4 , then our sum would be $16 - 12 - 4$ which is 0: not the desired answer!

If $16 - 12 - 4 \times (-1)$ is to equal 8, then $-4 \times (-1)$ must be $+4$.

So if multiplying out the brackets is to work, and mathematics is to be consistent, then the product of two negative numbers has to be positive.

Incidentally, consideration of negative numbers gives one clue to why it is generally preferable to put the numerical coefficient first in a term like $4N$. This convention makes it easier to cope with multiplying by negative numbers.

► What is the best way to write -4 multiplied by N ?

The convention of putting the number first gives $-4N$, which is fine. But if putting the number second was allowed, you might be tempted to write $N - 4$, which of course simply means 'N minus four' and is quite different. Of course, you could write $N(-4)$, to avoid the ambiguity; but if you adopt the 'number first' convention, the problem never arises.

Write a note in your own words why 'a minus times a minus is a plus' in your Learning File.

Dividing a number into a bracket

The process of multiplying out of brackets can be adapted to deal with the case of a bracket which is being divided by a number rather than multiplied. Here is an example:

$$(10X - 6) \div 2 \text{ is the same as } (10X \div 2) - (6 \div 2), \text{ which is } 5X - 3$$

Note that *each* term in the bracket must be divided by the 2.

The same principle is involved as when multiplying a bracket by a numerical factor. Indeed, if you consider that dividing by 2 is the same as multiplying by $\frac{1}{2}$, you will see that there is no essential difference between the two cases.

Activity 27 Getting rid of brackets: division

Express the following without the division sign and, where possible, without brackets.

$$(a) (4N + 8) \div 4 \quad (b) (6m + 3) \div 3 \quad (c) 5(4 - 2z) \div 2$$

This has probably drawn your attention to a similarity between multiplication and division in this context: there is an important difference too. Unlike the multiplication sign, the division sign cannot be dispensed with in algebraic expressions. Clearly, $(4N + 8) \div 4$ is not the same as $(4N + 8)4$, because in the latter expression the 4 then multiplies the bracket rather than divides it. However, there are alternatives to the use of the division sign when writing down expressions involving division. They are based on the various usual ways of writing fractions. Instead of $(4N + 8) \div 4$, you can write any of the following:

$$(4N + 8)/4$$

or

$$\frac{(4N + 8)}{4}$$

both of which are read as '4N plus 8 all divided by 4';

or

$$\frac{1}{4}(4N + 8)$$

which is read as 'a quarter of 4N plus 8'.

Note the importance of using brackets in the first of these alternatives: the expression you get if you leave them out, $4N + 8/4$, means $4N + 2$.

Algebraic fractions

This way of writing fractions extends to fractions involving algebraic expressions in both the numerator (top) and the denominator (bottom), which are called *algebraic fractions* or *quotients*. You could write $(4N + 8)$ divided by $(4N + 1)$, that is, the algebraic fraction $4N + 8$ over $4N + 1$, as:

$$(4N + 8)/(4N + 1)$$

or

$$\frac{(4N + 8)}{(4N + 1)}$$

or

$$\frac{4N + 8}{4N + 1}$$

All of these expressions are read as '4N + 8 all over 4N + 1'.

Leaving out the brackets, in this last case, does not cause any ambiguity; there is nothing else that the expression could mean except 'evaluate $4N + 8$ and $4N + 1$ and then divide the former by the latter'. The line dividing the numerator from the denominator acts, in effect, as brackets for each part of the fraction. It is essential to include the brackets when using the / version, however.

You could even write $\frac{1}{4N + 1}(4N + 8)$, read as 'one over 4N plus 1 times (pause) 4N plus 8'.

You may see the fraction written in either form; usually the version with the / sign is used in text, the 'built up' fractions in expressions which are displayed on a separate line.

Example 2 Evaluating an algebraic fraction

Evaluate $(6X + 3)/(4X - 1)$ when $X = 1$, and when $X = 4$.

When $X = 1$, $(6X + 3)$ is $6 + 3$ or 9, and $(4X - 1)$ is $4 - 1$ or 3.

So the value of the quotient $(6X + 3)/(4X - 1)$ is $9/3$, which is 3.

When $X = 4$, $(6X + 3)$ is $24 + 3$ or 27, and $(4X - 1)$ is $16 - 1$ or 15.

So the value of the quotient $(6X + 3)/(4X - 1)$ is $27/15$, which is $\frac{9}{5}$ or 1.8.

Think of a number revisited

To sum up this section, look back at the ‘Think of a number’ sequence from Subsection 1.1, with the hindsight that this extensive discussion of algebra provides.

The instructions went like this:

Frame 2

Think of a number		7
Double it	2×7	14
Add 5	$14 + 5$	19
Double the result	2×19	38
Subtract 2	$38 - 2$	36
Divide by 4	$36 \div 4$	9
Take away the number you first thought of	$9 - 7$	2

Take N to stand for the number you first thought of; then at the end of the sequence of instructions you have built up the expression:

$$(2(2N + 5) - 2)/4 - N$$

That looks quite complicated; but of course the reason that the ‘trick’ works is that in fact it is not complicated at all. A little algebraic manipulation shows why. The expression can be simplified by carrying out the following steps:

- (a) multiply out the inner bracket

$$(4N + 10 - 2)/4 - N$$

- (b) collect like terms (here, the numbers)

$$(4N + 8)/4 - N$$

- (c) divide out the bracket

$$N + 2 - N$$

- (d) collect like terms again (the N s this time)

$$2$$

That is to say:

$$(2(2N + 5) - 2)/4 - N = 2$$

So the basis of the ‘trick’ is to produce an expression which looks complicated, but actually unravels (when you simplify it) to a number which is independent of the number you first thought of.

If you have ambitions to be known as a mind-reader, you might like to construct a ‘Think of a number’ sequence for yourself.

Activity 28 *Think of a number again*

- (a) Think of a positive number; subtract 1; square the result; add twice the number you first thought of; subtract 1; take the square root; take away the number you first thought of; and your answer is 0. Explain.
- (b) Work out an expression for the end result of carrying out the following steps, if you call the number you start with X : think of a number; triple it; add 10; double the result; subtract 8.

Using your expression, complete the steps to produce a 'Think of a number' sequence.

- (c) Invent a 'Think of a number' sequence which begins 'Think of a number; quadruple it' and ends 'take away the number you first thought of, and your answer is 3'.

Afterword

When you are learning algebra, you naturally try to think what each expression represents, in more familiar terms. But one of the great labour saving benefits of algebra is that you do not really have to know what it all means when you are doing any calculations: you just apply the rules, confident that the interpretation will take care of itself. This is one reason why almost no contexts were provided in this section, why there were no examples indicating where these algebraic expressions had come from.

There is an English proverb 'take care of the pence, and the pounds will take care of themselves'. A reverse version might be 'take care of the sounds, and the sense will take care of itself'. This section has been about how to 'take care of the sounds'. When you get to this stage—which is like being able to think in a foreign language rather than having to translate what you want to say from your mother tongue—you really are speaking the language like a native speaker.

This point was well made by the seventeenth-century philosopher George Berkeley.

That there are many names in use among speculative men, which do not always suggest to others determinate particular ideas, is what nobody will deny. And a little attention will discover, that it is not necessary (even in the strictest reasonings) [that] significant names which stand for ideas should, every time they are used, excite in the understanding the ideas they are made to stand for: in reading and discoursing, names being for the most part used as letters are in *algebra*, in which though a particular quantity be marked by each letter, yet to proceed right it is not requisite that in every step each letter suggest to your thoughts, that particular quantity it was appointed to stand for.

(*'A Treatise concerning the Principles of Human Knowledge', Introduction, S19, 1734 in Michael Ayers (ed.) (1975) Berkeley's Philosophical works, J. M. Dent, (2nd ed)*)

Before completing this section, take some time to review what you have covered. Complete the Handbook activity sheet to give you a record of new ideas you have met. Include some examples if you find it helpful to do so. On your Learning File sheet, record the progress you are making. If it is helpful to you, use the section outcomes to measure your progress against. Do you feel you are confident in using the skills and techniques introduced in the section, or do you need more practice? Which activities can you cite as evidence of your progress in using algebra?

Outcomes

After studying this section, you should be able to:

- ◇ recognize whether or not collections of symbols are algebraic expressions (Activity 15);
- ◇ describe in your own words the meanings and use of the terms 'substitute', 'evaluate', 'value', and 'variable' (Activity 13);
- ◇ evaluate an algebraic expression by substituting a value for a variable (Activities 14, 19, 20);
- ◇ multiply or divide an algebraic expression by a number (Activities 22 and 27);
- ◇ multiply out brackets (Activities 20, 22, 23, 24, 25, 26, 28);
- ◇ explain in your own words why a negative number times a negative number is a positive number (Activity 13);
- ◇ simplify an algebraic expression (Activities 16, 17, 18, 20, 21).

3 Algebra and your calculator: formulas



Aims This section aims to teach you how to enter formulas into your calculator, and how to use your calculator to evaluate them. ◇

3.1 Mathematical functions

The things that a calculator will do, the tasks that it will carry out for you directly, are often called its ‘functions’. Some of a calculator’s functions deal with its mode of operation—the way it accepts and presents data, for example. Others are mathematical operations which you will find implemented in one way or another on practically any scientific calculator. For example, every scientific calculator has some way of calculating the square of any number: the x^2 function. In order to activate the x^2 function, you may have to press a key or select a command from a menu or carry out some other action—different calculating devices operate in different ways. These differences do not matter here. It is the function of calculating the square which is of interest.

There is something specifically mathematical about functions like the x^2 function, in contrast to those of the calculator’s functions which are concerned with its own mode of operation. The special nature of mathematical functions can be described as follows: a mathematical function is a processor, which takes input numbers, processes them according to some rule, and produces output numbers. The x^2 function does precisely that: you input a number; you do whatever you have to do to activate the process on your calculator; it delivers an output, produced according to the rule ‘square the input number’. The process is illustrated in the following diagram.

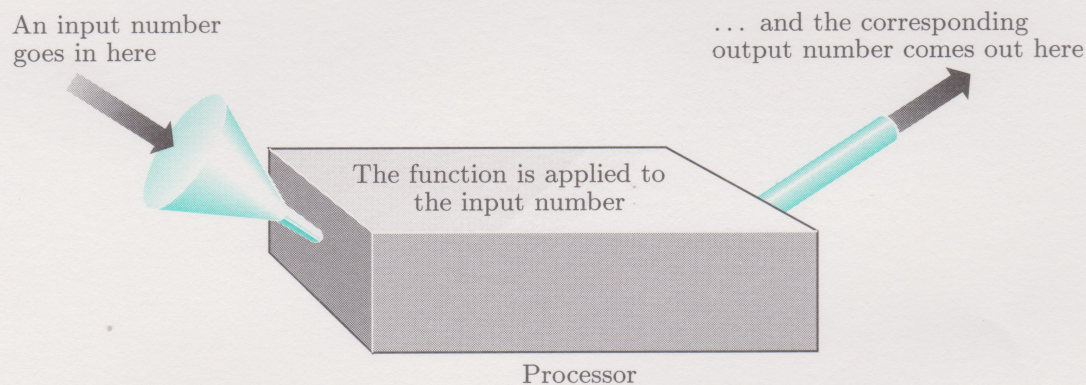


Figure 1 A function represented as an input–output machine

From the algebraic point of view, x^2 is just an expression, with x the variable. This process, described in terms of inputs and outputs, can equally well be described as the process of evaluating this expression for any given value of the variable. Choose another variable to stand for the output, say y ; the effect of the x^2 key is to take an input, x , and produce an output, y , according to the rule represented by the formula $y = x^2$. Any formula which can be represented algebraically works in the same way: it takes an input and processes it to produce an output according to the rule encoded in the formula. There is nothing special about $y = x^2$, apart from the fact that it occurs sufficiently often in practice for it to be worth a calculator manufacturer's while to implement it on any scientific calculator.

The purpose of this section could be described as showing you how the calculator makes up for the fact that it has not got specific ways of implementing all the other formulas you might want to evaluate.

A formula, therefore, specifies a mathematical function. You need to know the terminology of functions, as well as formulas, because both terminologies are commonly used in mathematics. But you should not find this change too difficult to cope with if you remember the analogy with the calculator's x^2 function.

Here are some more remarks about the use of terms and symbols when working with the calculator.

When doing algebra independently of the calculator, you have complete freedom in the choice of letters to name variables. The designers of the calculator could not anticipate which letters people would choose, however, so the calculator must impose some discipline in the use of symbols. In fact, most calculators' rules about which symbol is used are based on a decision (which is very frequently used in mathematics), which is to use x for the input variable and y for the output variable. This fits in with other occurrences of x and y with which you should be familiar, such as their uses as names of coordinates, and of axes when graphing. (There is one common exception to this convention, which arises most often when dealing with a problem in which some quantity varies with time: the symbol t is employed in such circumstances.) With this choice of letters, formulas always take the form:

$$y = \text{some expression involving } x$$

The variable y is called the *dependent variable*, because it depends on x (indeed, the formula shows exactly how it depends on x); x is called the *independent variable*. You met these terms in *Unit 7*.

A formula represents a function, in the form:

dependent variable = some expression in the independent variable

The standard choice of letters to stand for the variables is x for the independent variable, y for the dependent one.

In Activity 5, you worked out the function which takes as its input the measure of a temperature in degrees Celsius and gives as its output the measure of the same temperature in degrees Fahrenheit. In this case, the Celsius temperature is the independent variable, the Fahrenheit temperature the dependent one. It is natural to choose c for the independent variable and f for the dependent one, and to express the function as $f = \frac{9}{5}c + 32$. But if this function is to be entered into a calculator it usually must be written as $y = \frac{9}{5}x + 32$. This change of names makes no essential difference: the substance of the formula lies in the algebraic relationship it expresses, and this is not affected by the substitution of x for c and y for f . The only thing that changes is that there is no longer any obvious clue in the formula to how the variables are to be interpreted.

3.2 Using the calculator with formulas

In this section, you will learn how to enter and store formulas, or functions, on your calculator. Once a function is stored, you can do several different things with it. You can get the calculator to evaluate it for any chosen value of the variable (any number you think of) or you can get the calculator to draw up a table giving the values of the function for different values of the variable.

Work through Sections 8.1, 8.2 and 8.3 of Chapter 8 of the Calculator Book.



Outcomes

After studying this section, you should be able to:

- ◇ enter and store functions in your course calculator;
- ◇ use your calculator to evaluate a stored function for particular values of the independent variable;
- ◇ use your calculator to produce a table of values for a function.

4 A change of subject

Aims This section introduces some useful things you can do with formulas. ◇

4.1 Rearrangements

The formula which converts the measure of a distance in miles, M , to its measure in kilometres, K , is:

$$K = 1.61M$$

This is handy if you know how far it is from A to B in miles and you want to know how far it is from A to B in kilometres. It is not quite so handy if you know how far it is from α to β in kilometres and you want to know how far it is from α to β in miles. It appears that a new formula is required when the roles of the units of measurement are interchanged.

► What is the formula which converts the measure of a distance in kilometres, K , to its measure in miles, M ?

This is not very difficult; indeed, you have already found the answer in Activity 3; but it is worth spending some time thinking the problem through, because it proves to be the first simple step along what is rather an important path.

There are a couple of slightly different ways of arriving at the answer. The first is to work out what 1 kilometre measures in miles, and then say that to find what a distance of K km is in miles you just have to multiply this quantity by K . Now 1 mile is 1.61 kilometres, and 1 kilometre is $1/1.61$ miles, which is 0.62 miles (to two decimal places, as before), so:

$$M = 0.62K$$

This is a useful way of writing the formula; but it conceals the way the formula was derived, and that is the agenda of this section. It is better *for this purpose* not to work out what $1/1.61$ is, but instead to retain the *structure* of the calculation, and write the formula as:

$$M = \frac{1}{1.61}K$$

One further step: multiplying K by $1/1.61$ is the same as dividing K by 1.61, so:

$$M = \frac{K}{1.61}$$

Notice that the action has now moved to Greece, where distances are normally measured in kilometres, of course. Greek letters are perhaps less familiar to you, but they are still simply symbols. Recall α is the Greek letter 'alpha', and β the Greek letter 'beta'.

This way of finding the formula which gives M in terms of K is one which starts from scratch; it uses the same basic method that can be used to find the conversion formula between measurements of distance for any two units of length; and many other conversion formulas as well. It does not take account of the fact that you knew the formula for K in terms of M already. Yet there is a close relation between the two formulas, as you probably noticed. Just to make it clear, here they are written down, one above the other.

$$K = 1.61M$$

$$M = \frac{K}{1.61}$$

- How are the two formulas related? (This is a particular case of an important general principle, so it is worth your while to pause and think about the question, and not just rush straight past. The principle will be stated a few paragraphs below.)

Look at the same question—what is the formula which converts the measure of a distance in kilometres, K , to its measure in miles, M ?—from a slightly different point of view. You know the formula for K in terms of M ; how can you use it to obtain the formula for M in terms of K ? You know that $K = 1.61M$; you know how to use this formula to find the value of K when you are given the value of M . You are now faced with the problem of finding the value of M when you are given the value of K ; and, furthermore, of expressing the result of doing this as a formula giving M in terms of K . Now the required value of M is such that when it is multiplied by 1.61 the result is the value of K . The number that has this property is the value of K divided by 1.61; the required formula is therefore:

$$M = \frac{K}{1.61}$$

Activity 29 Pounds to kilos

The mass of something in kilograms, J , is given in terms of its mass in pounds, P , by the formula $J = 0.45P$. What is the formula which gives mass in pounds in terms of mass in kilograms?

Activity 30 Pounds to dollars

Conversions between currencies do not work in quite the same way. Since conversion rates fluctuate so much, this question is set in the distant past (the 1920s) when £1 (sterling) was worth \$4 (US). Then, the number of dollars (\$), S , a bank would give you for £ L was worked out by the simple formula $S = 4L$. But the number of £ the bank would exchange for \$ S is not given by $L = \frac{1}{4}S$ as you might expect: why not? Do you expect that the factor in front of the S in the actual formula to be greater than or less than $\frac{1}{4}$?

The word 'weight' is often colloquially used instead of 'mass'.

You have been considering formulas relating two variables. The examples discussed all take the form:

$$y = \text{some numerical factor} \times x$$

(Different letters have been used for the variables in different examples: here, just y and x were used.) Replace the phrase ‘some numerical factor’ by a single symbol, say c (for ‘constant’). So the examples discussed take the form:

$$y = cx$$

For the conversion of miles to kilometres, x is the measure of a distance in miles, y is the measure of the same distance in kilometres, and $c = 1.61$.

Activity 31 Interpretation

Interpret x , y and c for the conversion formula for masses in Activity 29.

The formula $y = cx$ gives y in terms of x ; the task is to convert it into a formula which expresses the same relationship, but gives x in terms of y . This is frequently expressed as follows: y is said to be the *subject* of the formula $y = cx$ (this makes grammatical sense: y is indeed the grammatical subject of the sentence ‘ y equals c times x ’). The task is to *rearrange* the formula so as to make x the subject instead of y . The formula giving x in terms of y is:

$$x = \frac{y}{c}$$

In words, to rearrange the formula $y = cx$ so that its subject becomes x , you *divide* by the factor c .

There are other kinds of formula which can be rearranged to produce a change of subject. One example is discussed in the following activities.

Activity 32 About my family

- (a) I was twenty-five years old when my daughter was born. So her age, y years, is given in terms of mine, x years, by the formula:

$$y = x - 25$$

Now let me retell the story from her point of view. She says: ‘When I was born, my father was twenty-five. So if I want to work out his age, knowing mine, I must use the formula ...’. Complete the sentence.

- (b) That was the personal interest version of the question. This is the dry mathematical version. Rearrange the formula $y = x - 25$ to make x its subject.
- (c) Tell the story the other way round and rearrange the formula $y = x - 25$ to make y its subject.

Activity 33 *What is the principle?*

There is a general principle at work here, too. Suppose you have a formula of the form $y = x + c$, where c is some number. How do you rearrange it to make its subject x ? Can you explain why you do it like that?

Activity 34 *Kelvin*

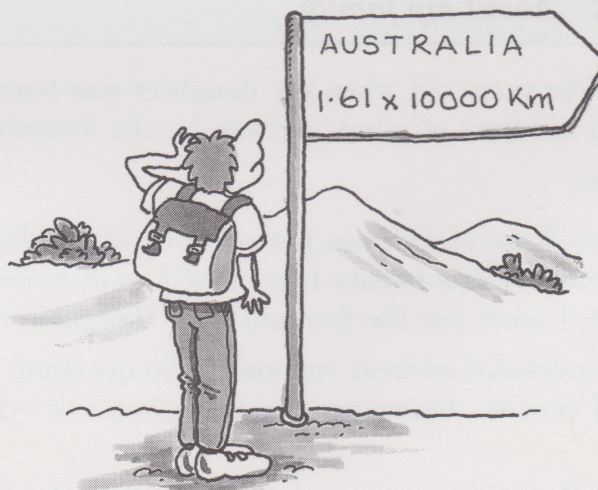
For many scientific purposes—for example, in low temperature experiments on superconductors—temperatures are measured on the absolute temperature scale; the unit in this scale is the Kelvin. The relationship between a temperature of k Kelvin and the equivalent Celsius temperature $c^{\circ}\text{C}$ is:

$$k = c + 273$$

The reason the number 273 appears in this formula is that -273°C is 0 Kelvin, the absolute zero of temperature (the temperature below which it is thought impossible to go).

What is the formula for converting a Kelvin temperature to a Celsius one?

A formula expresses a relationship between two variables. Formulas are usually expressed from one particular point of view: for example, the formula $M = K/1.61$ expresses the relationship between the kilometre and mile measures of a distance from the point of view of an insular Briton visiting, say, Australia. Rearranging the formula does not change the relationship that it expresses, only the point from which it is viewed. The formula $K = 1.61M$ expresses the same relationship, but from the point of view of, say, a forward-looking, internationally-minded Australian, visiting the UK and being unable to conceal his impatience with the national reluctance to measure distances the same way as everybody else.



Rearranging a formula involves changing its subject from one variable to the other. So far, this section has considered two types. The first is that to rearrange a formula of the form $y = cx$ in order to make x its subject, you divide out by the constant c to get $x = y/c$. The second is that to rearrange a formula of the form $y = x + c$ in order to make x its subject, you subtract the constant c to get $x = y - c$.

Here is yet another slightly different way of looking at this, which is very important. In both of these cases rearranging the formula involves undoing or reversing an operation. In the case of the formula $y = cx$, the operation in question is multiplication by c : the formula says ‘to obtain y , multiply x by c ’. To undo multiplication by c , you have to divide by c , and this is exactly what the rearranged formula $x = y/c$ says: ‘to obtain x , divide y by c ’. The second case is similar, but with multiplication replaced by addition. The formula $y = x + c$ says ‘to obtain y , add c to x ’. To undo addition of c , you have to subtract c , and this is exactly what the rearranged formula $x = y - c$ says: ‘to obtain x , subtract c from y ’.

Recall in this context the doing-undoing ideas and diagrams in Chapter 1 of the *Calculator Book*.

Multiplication and division undo each other, as do addition and subtraction.

If you think of algebraic rearrangement in these terms, you will be able to deal with a wider range of problems than if you just use the two rules as originally stated. You must exercise a little care, however. It would be tempting to say that if $y = 6 + x$ then y is obtained by adding x to 6, so the formula should be rearranged to give $y - x = 6$. This is in fact a perfectly valid way of rearranging the formula—but it does not achieve the desired effect of making x the subject.

Activity 35 Rearrangements

Rearrange each of the following formulas to make the second variable (for example, x in part (a)) the subject of it. As you do so, think about the manipulations you are using, and how you would justify your solutions.

- (a) $y = 25x$ (b) $Y = X + 7$ (c) $p = \frac{q}{81}$
 (d) $y = 6 + x$ (e) $m = \frac{2}{5}n$ (f) $u = v - \pi$

Thinking about rearranging a formula in terms of undoing an operation allows you to deal with the formula $y = x \div (\text{something})$ just as easily as with the formula $y = x \times (\text{something})$; and also it allows you to deal with the formula $y = x - (\text{something})$ just as easily as with the formula $y = x + (\text{something})$.

Here is a point of interest about one of those examples which will turn out to have all sorts of important repercussions later on. You perhaps rearranged $m = \frac{2}{5}n$ to get $n = \frac{5}{2}m$ by thinking of the first formula as $m = \frac{2}{5} \times n$, and undoing the multiplication by dividing m by $\frac{2}{5}$ (which is the same as multiplying it by $\frac{5}{2}$). But you could think of the formula $m = \frac{2}{5}n$ as involving two operations carried out one after the other: first multiply n by 2, and then divide the result by 5. You could then tackle

the rearrangement in two stages, like this: first undo the division by 5, by multiplying through by 5, which gives $5m = 2n$; and then complete the job by undoing the multiplication by 2, by dividing through by 2, which gives $n = 5m \div 2 = \frac{5}{2}m$, as before. (You could also think of the two operations which are applied to m to give n as being carried out in the opposite order: if you first divide by 5, and then multiply the result by 2, you will still get the right formula for n . Undoing the operations in the opposite order also produces the correct formula for m in terms of n , so everything fits together satisfactorily.)

The reason for dwelling on this particular case is that it gives a clue as to how to rearrange more complicated formulas which involve more than one operation. You do it as a sequence of steps, undoing one operation at a time.

Example 3 Hedge formulas

The formula which gives the age of a hedge in years, A , in terms of the number of species of hedgerow bush per 30 yards, N , is $A = 110N + 30$.

Now it is not uncommon to come across a reference to a hedge in a historic document, which might refer to a hedge that currently exists: one way of testing whether or not it does is to work out from the presumed age of the hedge how many species there should be, and then go to see whether the number of species now there agrees with the calculation. This is a situation in which it would be useful to rearrange the formula to make its subject N . How is this to be done?

Read the formula first, to make sure you understand what operations it involves. It says: to find the age of the hedge, multiply the number of species by 110 and then add 30 to the result. So there are two operations involved, multiplying by 110 and adding 30. Note carefully that the multiplication is done first, and then the addition. To make N the subject of the formula, you must undo these two operations in turn. The first operation to undo is the one which was last to be done. So the first step in rearranging the formula is to undo the addition of 30, that is, to subtract 30 to leave $110N = A - 30$.

You now have to undo the effect of multiplying by 110. That is to say, you have a formula of the form $110N = \text{something}$, and you want to rearrange this to give $N = \text{something (else)}$. So you must divide by 110, to obtain $N = (\text{something}) \div 110$; but be careful: it is the whole expression $A - 30$ that must be divided, not just A . The rearranged formula is thus:

$$N = \frac{A - 30}{110}$$

Or, if you prefer:

$$N = \frac{1}{110}(A - 30)$$

Now, about this reversal of order of operations. There is nothing very special about it: in fact, it is what you nearly always have to do when undoing any sequence of operations. You put on your socks or tights before your shoes, but you have to take off your shoes before you can remove your socks or tights. You probably run the bathwater before getting into the bath, but you get out of the bath before you let the water out. It is only when the order of doing things does not matter that the order of undoing them does not matter either. It does not usually matter whether you put on your left or right shoe first; and it does not usually matter which one you take off first, either.



In the case of the formula $m = \frac{2}{5}n$, first multiplying n by 2, and then dividing the result by 5 has exactly the same total effect as first dividing n by 5 and then multiplying the result by 2; so it does not matter in which order you do these things in when rearranging that formula. However, in Example 3, it does make a great deal of difference whether you first multiply N by 110 and add 30 to the result or whether you first add 30 to N and then multiply the result by 110. Thus, in the case of the hedge formula, you do have to be careful in which order you undo the operations.

Here is a way of thinking about rearranging the hedge formula which may help to clarify the steps for you. The second and final operation in constructing the formula is to add 30. For the moment, do not worry what it is you add the 30 to: just call it something, say M (algebra is such a liberating influence: you can call something like $110N$ anything you like, within reason!). The formula becomes $A = M + 30$. Rearranging this to make its subject M is easy: $M = A - 30$.

This is now the moment to remind yourself what M is: it is $110N$. You now have $110N = A - 30$. That is, $110N = \text{something}$; call this something B , so $110N = B$. Rearranging this to make N the subject is also easy: you get $N = B/110$. But B stands for $A - 30$, remember; so the final formula is $N = (A - 30)/110$ (note carefully the use of the brackets to make it clear that 110 divides the whole expression $A - 30$).

In every case, you need to know the order of the steps that were used to build up the formula and undo them in the reverse order. In this case, the formula is built up as follows.

Step 1: Multiply by 110 (to get $110N = A - 30$)

Step 2: Add 30 (to get $110N + 30 = A$, and hence $A = 110N + 30$)

To rearrange the formula, undo them in the reverse order.

Step 1: Subtract 30 (to get $A - 30 = 110N$ or $110N = A - 30$)

Step 2: Divide by 110 (to get $N = \frac{A - 30}{110}$ or $N = (A - 30)/110$)

The box below gives a summary of the processes that have been used to rearrange this formula, which should help to make it clear how to use them in other similar situations.

Starting off with a formula which gives y in terms of x (using standard symbols for the variables), the task is to rearrange it so that x is expressed in terms of y . The rearrangement proceeds by several steps.

At any stage, you will have an intermediate formula which takes the form:

some expression in y = some (different) expression in x

If the right-hand side is actually

(terms involving x) + c ,

then the next step in the rearrangement is to undo the addition of c , to give:

(expression in y) - c = some expression in x

For example,

$$y = \frac{3}{2}x + 5$$

$$y - 5 = \frac{3}{2}x$$

If, however, the right-hand side expression in x is actually

(terms in x) $\times c$,

then the next step in the rearrangement is to undo the multiplication of c , to give:

(expression in y) $\div c$ = some expression in x

Continuing with the example given above,

$$y - 5 = \frac{3}{2}x$$

$$\frac{2}{3}(y - 5) = x$$

Steps like these are repeated one after another until x by itself becomes the subject of the formula (usually no more than two or three steps are required).

You should notice, from this summary, that nothing more is involved than the simple steps required to rearrange the formulas $y = x + c$ and $y = cx$; the only differences are that you may have to carry them out on *expressions* in x or in y , and not just x and y by themselves; and you may have to carry out several such steps in succession.

Finally, it helps enormously if you are clear in your mind how the original formula is made up; the best way of doing this is to translate the formula into English.

Example 4 Temperature conversions

The formula which relates the measure of a temperature in degrees Fahrenheit, f , to the measure of the same temperature in degrees Celsius, c , is $f = \frac{9}{5}c + 32$. What rearrangement of this formula gives the temperature in degrees Celsius in terms of the temperature in terms of degrees Fahrenheit?

Translate the formula into words first, to make sure you understand what operations it involves. It says: to find the Fahrenheit measure of a temperature, you take the Celsius measure, first multiply it by $\frac{9}{5}$, and then add 32 to the result.

- Now, what is the first step in rearranging this formula? Remember, it is to undo the operation that comes last as far as the formula is concerned.

The last operation in the formula is the addition of 32; so the first step in the rearrangement is to undo that operation, which means undoing the addition of 32 to $\frac{9}{5}c$ by means of the subtraction of 32. The result is the intermediate formula $f - 32 = \frac{9}{5}c$ or $\frac{9}{5}c = f - 32$.

- What remains to be done to make c the subject of the formula?

You must now undo the multiplication of c by the factor $\frac{9}{5}$; this is done by dividing by $\frac{9}{5}$, which is equivalent to multiplying by $\frac{5}{9}$, to give the following formula for c in terms of f :

$$c = \frac{5}{9}(f - 32)$$

Activity 36 A different subject

In each of the following, make x the subject of the formula.

- (a) $y = 2x + 1$
 (b) $y = \frac{1}{2}x - 1$
 (c) $y = 2(x + 1)$

Activity 37 Kelvin again

Recall that the relationship between a temperature of k Kelvin and the equivalent Celsius temperature $c^{\circ}\text{C}$ is:

$$k = c + 273$$

- (a) What is the formula for converting a Fahrenheit temperature to a Kelvin one? (Use the formula from the end of Example 4 for c in terms of f .)
- (b) What is the formula for converting a Kelvin temperature to Fahrenheit?

Activity 38 Detective work

In the detective story 'The Ten Nailers' by Dorothy L. Sayers, there is an important clue to the crime which is concerned with the hire of a motorbike by one of the suspects. The detective, Lord Peter Fancy, sends his manservant Bunting to the Imperial Motorbicycle Hire Company, of London W1, to find the cost of hiring a motorbike. Bunting says: 'I have ascertained, my lord, that motorbicycles may be hired for the day at a cost of 5 guineas, to which must be added one shilling for every mile that the intrepid rider covers'.

- (a) Derive the formula which gives the cost in guineas of the hire of a motorbike for a day in terms of the number of miles ridden. (One guinea was 21 shillings.)

The plot hinged on whether the suspect could have hired the motorbike to travel from London to Norfolk to commit the crime when he claimed to have been in London at the time. Lord Peter finds the stub of a cheque made out to the Imperial Motorbicycle Hire Company for 12 guineas.

- (b) By rearranging the formula that you found in the first part of the question, work out how far the suspect had travelled on his hired motorbike that day.

4.2 Equations

In the final activity of the last subsection, you were asked to rearrange a formula in order to find the value of one variable when the value of another was given. The formula was $y = 5 + x/21$, and the problem was to find the value of x when y takes the value 12. You may have felt, while you tackled this question, that it was a bit unnecessary to find a general formula giving x in terms of y when all that is asked for is the value of x for one particular value of y . The problem that you were asked to solve could be expressed like this: find the value of x if $12 = 5 + x/21$.

When an expression like $5 + x/21$ involving just one unknown is set equal to some number, like 12, the result is called an *equation*. The problem is to find the value(s) of x for which the equation is satisfied: that is to say, such that substitution of any of these values for x does indeed lead to equality. A numerical value of x which satisfies the equation is called a *solution of the equation*, and the act of finding the required solution is called *solving the equation*.

In this case, the challenge is to solve the equation $12 = 5 + x/21$. The solution is actually 147 (this is the value of x found in solving the problem in Activity 38). To confirm that it is the solution, substitute 147 for x in the expression $5 + x/21$, to get $5 + 147/21 = 5 + 7$; and since $5 + 7 = 12$ it is clear that 147 is indeed the solution of the equation $5 + x/21 = 12$.

It is useful to know how to verify that a given value of x is the solution of an equation; but it is far more useful—and interesting—to know how to find the solution(s), or in other words, how to solve the equation. There are several ways of solving equations. One way, which involves the use of the calculator, is described later in the unit. Another way is sheer guesswork: try substituting different values for x until you find one which satisfies the equation. A moment's reflection, though, will convince you that this is not usually a great idea.

► If you do not believe this, try solving the equation $120 = 5 + x/21$ by trial and error.

Another way is to rearrange a formula to make the unknown quantity its subject: this is the method which was used to solve the equation $12 = 5 + x/21$ in Activity 38.

Example 5 How old is ...?

As you may recall, I was twenty-five years old when my daughter was born. To find her age, therefore, you just have to subtract twenty-five from mine. The other day my wife and I were turning out some old papers and we came upon a family photograph taken on one of Mary's birthdays. (Mary, of course, is the daughter in question.) In the photo, Mary is taking her first ride on her new bicycle, and her parents are looking on, rather anxiously. 'You've put on a lot of weight since then,' observed my wife to me. 'How old were you when that photo was taken? Let me see—Mary was eight when she had that bicycle for her birthday present.'

That is really much too simple a problem to require algebra to solve it: you probably were able to work out how old the 'I' in the photograph was straight away. But its being simple makes it a very suitable example for a first attempt at solving equations using algebra.

The problem is to find how old 'I' was when the photograph was taken; that is to say, the age is an unknown quantity (at least, until you have worked out what it is), so in the time-honoured fashion of algebra use a letter to stand for it. Let x denote my age when the photograph was taken. Then Mary's age when the photograph was taken is $x - 25$ (as was remarked above, to find her age you just have to subtract twenty-five from mine). But she was 8 when the photograph was taken; so:

$$x - 25 = 8$$

This is an equation for x ; if you can solve it, then you will have the answer to the question 'How old was 'I' when the photo was taken?'

The nineteenth-century mathematics educator Mary Boole described one essence of algebra as 'working with the as-yet-unknown'.

The equation says: If you take 25 away from the unknown number you get 8. Which number has the property that if you take 25 away from it you get 8? Why, $8 + 25$, of course. So if $x - 25 = 8$, then $x = 8 + 25 = 33$. The solution to the equation is 33, and 'I' was 33 years old when the photo was taken.

The principle used to solve the equation is the same as the principle which was used in rearranging formulas: the principle of undoing, or reversing, an operation. Here, the operation applied to x is subtracting 25; to find the value of x apply the reverse operation, namely adding 25.

That may appear to be the end of this particular story, but there is one more thing to mention: a simple way of checking that you have got the right answer when solving an equation and that you have not made a mistake with the algebra. The claim is that 33 is the solution of the equation $x - 25 = 8$. What this claim means is that substituting 33 for x in the two sides of the equation should give the same number. So it is possible to check that you have got the right answer by substituting 33 for x : $33 - 25$ is indeed equal to 8, so there is no mistake.

Whenever you have solved an equation, you should check that your solution is correct in this way, by substituting it into the equation and making sure that the equation is satisfied.

Activity 39 A visit from Geoff

I had a visit from an Australian colleague, Geoff, recently—the one who is so impatient about the fact that the UK has not converted measurements of distances from miles to kilometres yet. I invited him to my house one evening; he wanted to know how far it is from where he was staying. I told him 12 miles; but of course he insisted that I tell him the distance in kilometres. The way I remember the conversion factor is this: one kilometre is about five-eighths of a mile.

(a) Let the distance to my house be x kilometres. Write down the equation that x must satisfy.

(b) How far is it to my house, in kilometres?

Do not forget to check your solution.

Example 6 Deriving a solution

You have already solved the equation $12 = 5 + x/21$ which arose at the end of the last subsection. You know what the solution is: 147. The question now is: can you derive it directly?

You can solve this equation by carrying out just the same steps that you did when you were rearranging formulas to change the subject. Indeed,

one good way of thinking about how to use algebra to solve an equation like this is that you must rearrange the equation so as to make x its subject. To make x the subject, after all, is to rearrange the equation so that it takes the form $x = \text{some number}$; and then that number is the solution of the equation.

But before starting, perhaps give some thought to the way the equation is presented.

- Does it matter that the unknown, x , appears on the right-hand side of the equation rather than the left?

No: if two things are equal they are equal; you can express equality as well by saying that the first thing is equal to the second as by saying that the second thing is equal to the first. If you prefer it, therefore, you can write the equation as $5 + x/21 = 12$: this expresses exactly the same relationship as the original version.

- Does it matter that the term containing x is to the right of the plus sign?

No: the sum of two things is the sum of two things; you can express addition as the first thing plus the second or as the second thing plus the first. So you can also write the equation as $x/21 + 5 = 12$: this again expresses exactly the same relationship as the original version.

So you can write the equation to be solved as $x/21 + 5 = 12$. Rearranging this equation to make x its subject involves two steps. First, subtract the 5, to give:

$$x/21 = 12 - 5 = 7$$

Then multiply by 21, to give:

$$x = 7 \times 21 = 147$$

The solution is therefore 147.

Finally, you should check the solution by substitution—but you have done this already.

Although not strictly necessary, it is often useful to turn the equation round before starting. If you are not familiar with a process like solving equations, it is usually best to standardize things as far as possible to begin with.

Activity 40 *An equation to solve*

By adapting the solution of the example above, solve the equation $120 = 5 + x/21$. (Do not forget to check your solution.)

Activity 41 *More equations to solve*

Solve the following equations.

(a) $2x + 1 = 5$

(b) $\frac{1}{2}x - 1 = \frac{1}{2}$

(c) $7x - 3 = 0$

(Do not forget to do what you must not forget to do.)

Activity 42 *Odd one out*

Which of the following is the odd one out?

$1 + 3x = 7 \quad 7 = 3x + 1 \quad 3x + 7 = 1 \quad 7 = 1 + 3x$

Problem 24*A quantity and its $\frac{1}{7}$ added together become 19. What is the quantity?**Assume 7.*

$\backslash 1$	7
$\backslash \frac{1}{7}$	1
Total	8.

As many times as 8 must be multiplied to give 19, so many times 7 must be multiplied to give the required number.

1	8
$\backslash 2$	16
$\frac{1}{2}$	4
$\backslash \frac{1}{4}$	2
$\backslash \frac{1}{8}$	1
Total $2 \frac{1}{4} \frac{1}{8}$.	

$\backslash 1$	$2 \frac{1}{4} \frac{1}{8}$
$\backslash 2$	$4 \frac{1}{2} \frac{1}{4}$
$\backslash 4$	$9 \frac{1}{2}$

<i>Do it thus:</i> The quantity is	$16 \frac{1}{2} \frac{1}{8}$,
$\frac{1}{7}$	$2 \frac{1}{4} \frac{1}{8}$,
Total	19.

In the third multiplication, instead of multiplying 7 by $2 \frac{1}{4} \frac{1}{8}$, the author multiplies $2 \frac{1}{4} \frac{1}{8}$ by 7, this being easier.

Problem 24 from the Rhind Papyrus

Historical note

Solving problems expressible as equations is an activity with a very long history. The Rhind Papyrus dates from about 1650 BC. It is one of our main sources of information about arithmetic and other mathematics in ancient Egypt. The Rhind Papyrus seems to have been a prototype course unit; at any rate, it contains a number of problems, some of which, from a modern point of view, require the solution of an equation. The Egyptians did not use algebra as we know it today. The particular solution is given, together with the numerical answer, but no general methods or discussion appears. Instead, the unknown quantity is referred to as 'aha', which is translated as 'heap'. For example, Problem 24 requires the student to find the value of heap if heap and a seventh of heap is 19. (It is hard to imagine this problem causing a great deal of excitement on the banks of the Nile.)

Activity 43 A problem from ancient Egypt

To show that modern methods can deal with ancient problems, solve Problem 24 of the Rhind Papyrus.

A heap and one-seventh of a heap gives 19. What is a heap?

Activity 44 A cold problem

The conversion formula from Fahrenheit to Celsius is $c = \frac{5}{9}(f - 32)$. What is the Fahrenheit equivalent of a temperature of -40°C (that is, 40 degrees below zero)?

A temperature of -40°F is the same as -40°C : an odd and unexpected result. Suppose that, in a mood of idle speculation, you were to ask yourself: 'Is there another temperature that is the same number in Fahrenheit and Celsius?' Is there some way of tackling such a question? Trial and error, or being told the answer in a course unit, are not regarded as satisfactory *methods*.

There is an unknown quantity, so call it x . The best way of saying what the number x represents is this: $x^\circ\text{F}$ is the same temperature as $x^\circ\text{C}$. So if you find the Celsius equivalent of $x^\circ\text{F}$, the answer you get must be equal to x . The Celsius equivalent of $x^\circ\text{F}$ is $\frac{5}{9}(x - 32)^\circ\text{C}$. Thus, x must satisfy the relationship:

$$\frac{5}{9}(x - 32) = x$$

Here is something which certainly deserves to be called an equation for x , but it is rather different from the equations encountered so far. Before,

each equation involved an expression in x set equal to a number. Now you have to deal with equations in which one expression in x is equal to another expression in x . So the notion of an equation has to be a bit more flexible than has been indicated so far. But this does not really change what it means to be a solution of an equation: it is a value of x which satisfies the equation. The test that a solution has to meet is this: when it is substituted for x wherever x appears in the equation, the resulting numerical values of the expressions on each side of the equation are equal.

Activity 45 Check

Confirm that -40 is a solution of the equation $\frac{5}{9}(x - 32) = x$; and (so that you see both sides of the coin) verify that 40 is *not* a solution.

Perhaps it would be wise to start with a rather less complicated example. Conundrums about people's ages are very popular as brain teasers. Here is one that leads to an equation which is similar to but not quite as complicated as the one about temperatures.

Example 7

I am twenty-three years older than my daughter (this is my other daughter, the elder of the two). She is quite grown up now. A year or two ago, I realized on her birthday that my age was then exactly double hers. How old was she when I made this discovery?

It is not very difficult to spot the answer: but the point of the exercise is to obtain it systematically. To do so, use the information given in the question to formulate an equation which the unknown quantity must satisfy—my daughter's age when I realized I was twice as old as she was. Then solve the equation.

Use x to denote the daughter's age when I realized I was twice as old as she was. Now I am twenty-three years older than she is, so my age then was $x + 23$. On the other hand, my age was twice hers, so my age then can also be expressed as $2x$. There are now two expressions for the same thing—my age when I realized that I was twice as old as my daughter; so these two expressions must be equal. Thus:

$$2x = x + 23$$

This is the equation which you can use to find the value of x .

So the remaining task is to solve the equation. A productive way of thinking about this task is to rearrange the equation so that it takes the simple form $x = \text{some number}$; in other words, make x the subject of the equation. The rules for rearranging a formula allow you effectively to move a term from one side to the other by undoing an operation of addition,

subtraction, multiplication or division. In most cases, you have moved a number from one side to the other; but there is also the possibility of moving a variable. You could, for example, rearrange the equation: the right-hand side was obtained by adding 23 to x , and undoing that operation leads to:

$$2x - 23 = x$$

But although this is a correct application of the rules, you are no further forward in the task of making x the subject. On the other hand, you could read the right-hand side the other way round, as the addition of x to 23; undoing this operation gives:

$$2x - x = 23$$

and this is much more useful. Since $2x - x = x$, the solution is:

$$x = 23$$

So the same basic rules that were used for rearranging formulas can be pressed into service in the solution of more general equations too. The extra ingredient is that it will often be necessary to move variables around, not just numbers. The art of solving equations using algebra is to use the rules of rearrangement to get the equation into the form $x = \text{some number}$. This may take several steps; a sensible intermediate aim is to rearrange the equation so that you have only multiples of x on one side, and only numbers on the other. The problem of the identical temperatures is a good case in point.

Example 8 *Fahrenheit = Celsius?*

Which temperature is the same in the Fahrenheit and Celsius scales?

If $x^\circ\text{F}$ is the same temperature as $x^\circ\text{C}$, then, as you have already seen, x must satisfy the following equation.

$$\frac{5}{9}(x - 32) = x$$

The task now is to solve this equation. First, try to get all the multiples of x on one side of the equation, and all the numbers on the other. The best way of doing this will be to collect the multiples of x on the right-hand side (because there are no numbers there, currently). But the term in x on the left-hand side is bound up inside a bracket; and before it can be moved, it must be 'set free'. So the very first step must be to multiply out the bracket on the left-hand side. This gives:

$$\frac{5}{9}x - \frac{5}{9} \times 32 = x$$

The term $\frac{5}{9}x$ can now be subtracted from both sides to get:

$$-\frac{5}{9} \times 32 = x - \frac{5}{9}x$$

Now $x - \frac{5}{9}x = \frac{4}{9}x$, so the above equation may be written as:

$$-\frac{5}{9} \times 32 = \frac{4}{9}x$$

or

$$\frac{4}{9}x = -\frac{5}{9} \times 32$$

in order to finish up with ' $x = \text{some number}$,' rather than ' $\text{some number} = x$ '. The final step is to undo the multiplication of x by $\frac{4}{9}$, which gives:

$$\begin{aligned} x &= -\frac{9}{4} \times \frac{5}{9} \times 32 \\ &= -\frac{5}{4} \times 32 = -40 \end{aligned}$$

The algebra has shown that -40°F is the same temperature as -40°C .

There are other routes to the same answer: for example, clearing the fraction first could be a good move, as follows. From the original equation, $\frac{5}{9}(x - 32) = x$. Multiplying by 9 gives:

$$5(x - 32) = 9x$$

Now multiply out the bracket, which leads to:

$$5x - 160 = 9x$$

Subtract the term $5x$ from both sides, to get:

$$-160 = 4x$$

and finish off by dividing through by 4.

But however you carry out the process in detail, the strategy is always the same: get the x s together on one side of the equation, the numbers together on the other side; collect terms; and finally divide by the coefficient of x .

You should do some practice now. There are really two separate stages in solving problems involving equations. One is the formulation of the equation; the other is its solution. You should probably practise each of these two stages separately, before you try the whole process for yourself. The following set of activities offers practice in solving equations first, then in formulating them, and finally in solving problems from scratch. One comment about these problems: when you get past the first activity, you will find that the problems are of the brain teaser kind. This is not to be taken to mean that algebra is just a game. Far from it: it is a very important tool in solving serious problems, as you will see often enough as the course continues. But brain teasers offer opportunities for you to practise formulating and solving equations which are suitable at this early stage in your study of algebra.

As you are working on these activities, think about the rules that you are using to solve the problems, and the reasons why those rules apply. Do not forget to check your answers (*before* turning to the solutions at the end of

the unit). It is not enough just to read about algebra if you are to learn how to use it. You must practise the skills for yourself. In fact, reading by itself is a very inefficient way of learning to do mathematics. This is why it is important for you to try to do the activities for yourself, and not just consult the answers.

Make sure that you have added notes on the new techniques you have learned in this section to your Handbook.

Activity 46 Solving equations

Solve the following equations.

- (a) $3x = x + 4$
- (b) $x + 1 = 2x - 1$
- (c) $3(x + 1) = 2x - 1$

Activity 47 Formulating equations

- (a) At 10 am, Alastair was 67.5 miles from Cardiff, heading towards Edinburgh at 45 mph, while Bethan was in Edinburgh, 400 miles from Cardiff, poised to start heading towards Cardiff on the same road at 50 mph. As you will no doubt remember, this means that t hours after 10 am, Alastair's distance from Cardiff (in miles) is $67.5 + 45t$, while Bethan's distance from Cardiff is $400 - 50t$. Write down an equation which would enable you to determine the time at which they meet.

Check that they meet at 1.30 pm.

- (b) Janet is ten years older than Leroy. In two years' time, she will be twice as old as Leroy. Formulate an equation which will enable you to work out how old she is now.

Check that Janet's age is eighteen.

- (c) In my house there are just as many (four-legged) pets as there are (two-legged) people. If I multiply the number of heads by the number of tails I get the number of feet. Find an equation for the number of people (or equivalently pets) in the house.

Check that I have three pets.

Activity 48 A Shakespearian problem

Shakespeare memorably described the seven ages of man in *As You Like It*, act II, scene vii.

Suppose that a man spends a fifteenth of his life as an infant, 'mewling and puking in the nurse's arms'; two-fifteenths of his life as a schoolboy

'creeping like a snail unwillingly to school'; a twenty-fifth as a lover; a fifth as a soldier; two-fifths as a justice; ten years as a 'lean and slipperd pantaloons'; and that the final two years of his life are spent in 'second childishness and mere oblivion; *sans* teeth, *sans* eyes, *sans* taste, *sans* everything'. What is a man's life expectancy?



Activity 49 Reviewing progress

Before moving on, take some time to review your learning and progress on this unit. Check over the Handbook sheet and your Learning File sheets and add any further notes. Think about what you have been learning. How well do you feel you can use the techniques you have just been practising? Has any aspect been particularly difficult? What feedback have you used to help you (for example, you may have been working with other students or your tutor, or used the comments in the unit yourself)? How does feedback help you in your learning?

Outcomes

After studying this section, you should be able to:

- ◇ change the subject of a formula (Activities 29 to 38);
- ◇ solve an equation algebraically in simple cases and check the solution (Activities 39 to 48).

5 Algebra and your calculator: graphs

Aims This section aims to teach you how to use your course calculator to produce graphs representing functions, and how to use the graphs you obtain to solve equations. ◇



5.1 Using the calculator to draw graphs

From Section 3, you should know how to enter and store a function (or formula) in your calculator, how to use the calculator to evaluate it for any chosen numerical value of the variable, and how to draw up a table evaluating the function for a number of different values of the variable. The next step is to get the calculator to draw a graph of a function. Following that, you will be shown how to use the calculator to explore the features of a graph, and in particular how to use the graphical facilities of the calculator to solve equations.

The process of drawing a graph to represent a function (or, as it was called there, a mathematical relationship) was described in Subsection 1.3 of *Unit 7*. For each value of x , the corresponding value of y is calculated using the formula which specifies the function, and then the point whose coordinates are (x, y) is plotted. The collection of points plotted in this way forms the graph. If you are drawing a graph by hand, on graph paper, you can plot only a limited number of points, and to obtain the graph you have to join them up smoothly. In many of the cases considered in this section, the result is a curve, not a straight line.

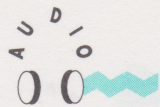
Work through Sections 8.4 and 8.5 of Chapter 8 of the Calculator Book.



5.2 Zooming around the course calculator

Altogether, four new calculator skills have been covered in this unit: *entering and storing functions*; *creating tables of values*; *drawing graphs*; and *solving equations* graphically using the *trace* and *zoom* facilities. Individually, these are important skills, but in many situations you need to use most or all of them one after the other. This makes quite a demand on your memory. It seemed likely that the first time you were asked to do an exercise which involves all these skills, you might welcome some help, so there is an audiotape sequence to assist you. In the following activity, you will be asked to enter, tabulate and graph quite a complicated function, and solve an equation by finding where the function takes the value zero. The speaker will talk you through the whole activity. In addition to the cassette and a tape player, you will need only your calculator, though you

might find it useful to have a piece of paper and a pen or pencil to hand, and to have written down the formula for the function, so that you do not have to keep referring back to the unit.



Activity 50 Trace and zoom

Use band 5 of Audiotape 2 to help you graph the function:

$$y = 1 - \frac{3x}{x^2 + 1}$$

Use your graph to solve the equation:

$$1 - \frac{3x}{x^2 + 1} = 0$$

5.3 Things to do

Activity 51 Conversion tables

If a distance is M miles and K km, then M and K are related by the formula:

$$K = 1.61M$$

Use your calculator to produce tables giving the kilometre equivalents of distances in miles, and also produce conversion graphs, suitable for the following purposes:

- use by someone familiar with the metric system but not the Imperial one, who is touring the UK by train, bus or car;
- use by a parish or town council for converting the distances given on signposts for footpaths and cycle tracks into kilometres.

You have now seen three ways of representing the process of converting from measurements of distances in miles to the corresponding measurements in kilometres: by means of the formula $K = 1.61M$; by means of the table you drew up in Activity 51; and by means of a conversion graph (which you first plotted in Subsection 1.2 of *Unit 7*). This is a good point to pause and compare these different methods of representing such relationships.

Each of the three representations of the conversion (symbolic, numerical and graphical) is useful in its own way; but each has its advantages and disadvantages, according to the context in which, and the purpose for which, it is being used.

The formula is very accurate, in principle: the only limits on its accuracy derive from the accuracy with which the conversion factor (1.61 in our formula) is known, and the accuracy with which calculations can be carried out. Furthermore, there is no limit, in principle, to the sizes of the distances you can convert. The formula can as easily be used for very small distances—two or three miles—as for very large distances, such as thousands of miles. However, using a formula can be time-consuming, especially if you have a large number of distances to convert.

A table is quick to use, but it can be used directly to convert only those distances which actually appear in it. If you want only approximate answers—if distances to the nearest ten miles are good enough, say—the table is very useful.

Like a table, a graph also covers only a specific range of distances. However, you can use a graph, unlike a table, to convert intermediate distances: those distances which do not appear in the table. In fact, you can convert other distances very accurately using the trace facility on your calculator. The graph has the added advantage of showing pictorially how miles and kilometres are related. Of course, the formula reveals the nature of the relationship just as clearly—but you do have to be able to read and understand formulas before you can appreciate them in this way.

So each representation has its own strengths, and is useful in appropriate circumstances.

Activity 52 *Cube root*

In this activity, you will use your calculator to calculate the *cube root* of 5. The cube root of 5, written $\sqrt[3]{5}$, is that number whose cube is 5: in other words, it is the solution of the equation $x^3 = 5$. The method to use here to calculate $\sqrt[3]{5}$ is to graph $y = x^3$ and $y = 5$ simultaneously, and find the x -coordinate of the point where the graphs intersect.

Store the functions $y = x^3$ and $y = 5$ in your calculator, and graph them both. You will find that the graph of $y = 5$ is a horizontal line, which crosses the curve representing $y = x^3$ near $x = 1.6$. Use the trace and zoom facilities to find the x -coordinate of the point of intersection to two decimal places.

You can check your answer by calculating its cube. Alternatively, you can calculate $\sqrt[3]{5}$ directly on your calculator.

Activity 53 *The problems of Judy*

Your friend Judy, you may remember, needs to hire a car from Monday to Friday. She has to choose between hiring it at the weekly rate of £235, which includes unlimited mileage, or the daily rate, where the cost depends on how far she drives. The cost in pounds of hiring the car for that period, say p , at the daily rate, driving m miles, is

$$p = 193.75 + 0.19m.$$

You last met Judy in Activity 10.

Use the calculator to estimate the mileage above which the weekly rate is more economical than the daily rate. (You can do this with a table or with a graph: use whichever method you prefer.)

Outcomes

After studying this section, you should be able to:

- ◇ use your calculator to produce graphs and tables of functions over a given range of the independent variable (Activity 50 and 51);
- ◇ use your calculator to solve equations graphically (Activities 50, 52 and 53).

6 Some applications

Aims The aim of this section is to discuss some applications of ideas you have been taught in the previous parts of the unit. ♦

The section ends by examining a problem which arises in medicine: how should an adult drug dosage be scaled down to a safe level when the same drug is to be administered to a child? But it prepares the ground by discussing some other applications of algebra.

6.1 Formulas for areas and volumes

There are many useful rules for calculating areas and volumes, of rectangles, circles, cylinders, cones, and so on. These formulas can be conveniently expressed algebraically.

For example, the area enclosed by a rectangle is the product of the lengths of its sides. This can be expressed as a formula as follows: if the sides of the rectangle are a and b , then its area A is given by:

$$A = ab$$

A word of clarification about units of measurement: assuming that a and b are the lengths of the sides in the same units of length, the formula will give the area in the corresponding units of area. If a rectangle measures a metres by b metres, its area will be ab square metres; if its sides are a inches and b inches, on the other hand, ab will be its area in square inches. The same formula works whichever units of measurement are used, provided you stay within one and the same system. The same is true for all the examples treated in this subsection.

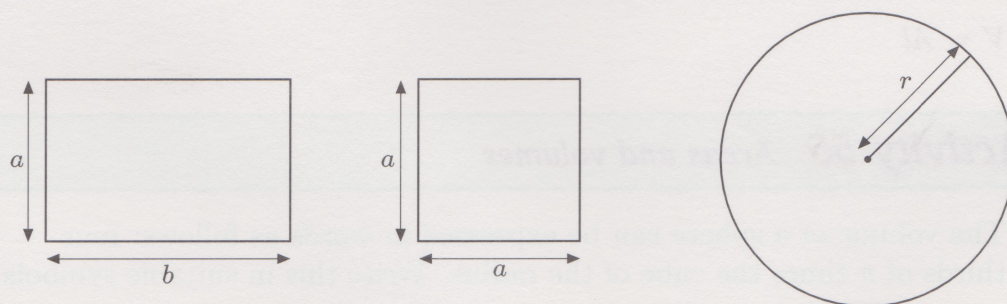


Figure 2 Rectangle, square, and circle

If the rectangle has equal sides—in other words, if it is a square—then its area is $a \times a$, which of course is written as a^2 and called ‘ a squared’.

The circumference of a circle of radius r is $2\pi r$, and the area enclosed by a circle of radius r is πr^2 .

Activity 54 Garden design

Suppose that you were planning a garden, and your plans included a lawn consisting of a rectangle 4.2 metres by 3.6 metres plus two half-circle pieces of diameter 3.6 metres added on to the shorter sides. Draw a plan of the lawn, and use the formulas above to estimate how much turf you would need to cover the area, giving your answer in square metres.

The volume V of a rectangular box of sides of length a , b and c is given by:

$$V = abc$$

You could think of this as the area of one rectangular face of the box, $A = ab$, multiplied by the other side, c :

$$V = Ac$$

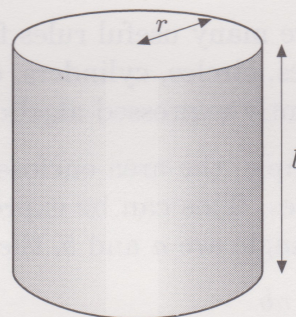
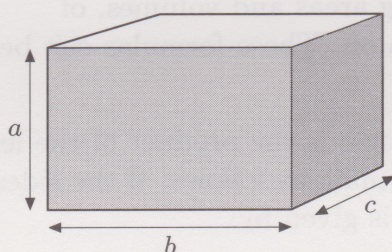


Figure 3 Rectangular box (showing sides and faces) and cylinder

The volume of a cube of side a is $a \times a \times a$, which is written a^3 and read as 'a cubed'.

The volume V enclosed by a cylinder of radius r and length l is given by:

$$V = \pi r^2 l$$

Alternatively, you might care to think of this as the area of the base $A = \pi r^2$ multiplied by the height l :

$$V = Al$$

Activity 55 Areas and volumes

- The volume of a sphere can be expressed in words as follows: four thirds of π times the cube of the radius. Write this in suitable symbols.
- A cylinder has radius r and length l . Its total surface area can be regarded as made up of three parts: two circular ends and a rectangle of length equal to the length of the cylinder and of width equal to the circumference of the circles at each end. (To see this, think about cutting the cylinder vertically and unrolling it flat.) Write down a formula for this total area in terms of r and l .

Activity 56 Party time

- (a) Sujatha has been invited to a fancy dress party, and wants to go as a can of fizzy lemonade. She has an ideal piece of strong, silver-painted card from which to make the main part of her costume, together with several smaller pieces which will do for the top and bottom (these will need to have holes in for her head and legs). She plans to make a can of height 0.8 m and with a radius of 0.2 m. What area of card will she need for the curved part of her costume? The piece of card she has is a rectangle 1 m by 1.1 m. Is it big enough?
- (b) Niels is going to a different fancy dress party. He is planning to go as a red UK cylindrical post box. Poor Niels does not have any suitable materials, and will have to buy some plain card and then paint it red. His costume will be a cylinder of height about 1 m and radius about 0.25 m; the post box will have a top but no base. He needs to know how much card to buy, and how much paint. What is the area of the outer surface of his post box?

The card comes in the following sizes: 1 m square, 1 m by 1.5 m, 1.5 m square, 1 m by 2 m, and 1.5 m by 2 m. Which should he buy? A small tin of supercover red paint will cover at least two square metres. Is this enough for Niels to paint the outside of his post box?

6.2 Ants and elephants

Have you ever wondered why big animals have such large feet? Disproportionately large, that is, by comparison with small animals. Why isn't an elephant just a scaled up version of an ant? Why are its proportions so different? In particular, why are an elephant's feet so much bigger, in relation to the rest of its body, than an ant's?

The answer depends on the fact that when you increase the dimensions of a body, its volume, and hence its mass, increases at a different rate from that at which its surface area increases. This is one example of the many problems which require an understanding of how areas and volumes, and other related quantities, are affected by changes of scale.

An elephant is roughly a thousand times bigger than an ant. Actually, since the whole point of this discussion is that degrees of bigness differ according to whether you are talking about mass, overall length or shoe size, this means that an elephant is a thousand times *as long as*, or *as tall as*, an ant. It is a thousand times larger in any typical *linear* dimension. The elephant's length is obtained from the ant's *by scaling it up by a factor of one thousand*.

- How are areas and volumes scaled up when a length is scaled up by such a factor?

Bodies and feet are odd shapes, but they could be approximated by cylinders and rectangles to give an idea of how a scaled-up version of an ant would compare with an elephant.

Begin the investigation by examining what happens when you scale up by a more modest factor.

► Suppose that you double each of the sides of a rectangle: how does this change its area?

If the sides were originally a and b , then when they have been doubled they become $2a$ and $2b$. The new area is:

$$2a \times 2b = 4ab$$

Thus, doubling the sides of a rectangle has the effect of multiplying its area by four.

Activity 57 Double double

What happens to the area of a circle if you double its radius?

What happens to the total surface area of a cylinder if you double both of its dimensions?

What happens to the area enclosed by a general figure if you double all its dimensions? What if you halve all its dimensions? What if you multiply all of its dimensions by ten?

What happens to the volume enclosed by a surface if you double all its dimensions? What if you halve all its dimensions? What if you multiply all of its dimensions by some fixed factor?

These various investigations suggest that if you double the linear dimensions (sides, radii, and so on), then areas get multiplied by 4—the square of the factor 2—while volumes get multiplied by 8—the cube of the factor 2. More generally, if you scale the linear dimensions by a factor k , then areas get multiplied by the factor k^2 , and volumes by the factor k^3 .

In the case of the ant and the elephant, the factor by which the ant's linear dimensions are scaled is about 1000. The area of the ant's feet would thus go up by a factor of 1000^2 , which is 1 000 000. However, its volume, and hence its mass, would go up by a factor of 1000^3 , which is 1 000 000 000.

Now the pressure that an animal experiences on the soles of its feet when it is standing or walking is its mass divided by the area of the soles of its feet. Consider what this means for the scaled-up ant. The area of the soles of its feet is 1 000 000 times that of an ordinary ant; the mass of the scaled up ant is 1 000 000 000 times that of an ordinary ant; so the pressure on the soles of the feet of the scaled-up ant has increased by a factor of

$$\frac{1\,000\,000\,000}{1\,000\,000} = 1000.$$

Thus, it would experience a pressure 1000 times greater than the pressure on the soles of the feet of an ordinary ant. It would have to have pretty thick skin (and strong bones) to take that strain!

So an ant scaled up a thousand times would have to have feet (and also legs) which could withstand a thousand times greater pressure than those of an ordinary ant. It is not surprising that insects survive with very thin legs and minute feet in proportion to the size of their bodies, but that elephants cannot. If an elephant's feet could withstand only similar pressure to those of an ant, the total area of the soles of its feet would have to be a factor 1000 times greater than those of the scaled-up ant.

Now ants have 6 feet, while elephants have only 4; so the area of each of the elephant's feet would have to be $1000 \times (\frac{6}{4})$ times greater than the area of each of the scaled up ant's feet; this is 1500. For the foot *area* to be 1500 times greater, the foot *length* must be $\sqrt{1500}$ times greater; $\sqrt{1500}$ is just under 40. So on the basis of this discussion you might expect the length of an elephant's feet, as a proportion of the overall length of its body, to be about 40 times bigger than the corresponding dimension of an ant.

Elephants do have proportionately bigger feet than ants, though not by quite such a large factor as this analysis suggests. In fact, they are nearer 10 times bigger rather than 40 times, as you can tell from the picture below, which shows an elephant and an ant scaled so their bodies are shown the same length.

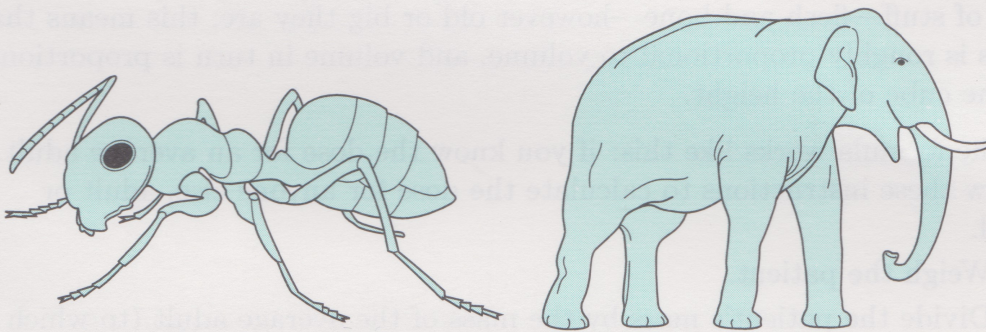


Figure 4 The ant and the elephant

The fact that the ratio is smaller than calculated suggests that elephant's legs and feet are rather stronger than the ant's. This is not surprising, because elephants, unlike ants, have a bone skeleton which strengthens their legs and feet.

Even though this simple-minded analysis did not give quite the right numerical answer, it does provide an interesting example of how you can understand how animals' shapes are affected by their sizes, as a consequence of the most basic facts of geometry.

6.3 Calculating drug dosages

The fact that volumes and areas are affected differently when you vary linear dimensions has all sorts of interesting repercussions. Here is one: it has to do with how you work out drug dosages for children.

It is not a good idea to give a child the same size dose of a drug as you give to an adult. How much of a drug you give someone—adult or child—will depend on how big they are. The rate at which the body clears a drug from the blood stream depends broadly on the surface area of the kidneys. So when someone is receiving a drug that has to be kept at a certain concentration in the blood stream, the dosage has, in effect, to be matched to the surface area of the patient's kidneys. This, of course, is not something that can be measured directly. But it is a reasonable guess that if one person (an adult) is twice as tall as another (a child), the surface area of the adult's kidneys will be four times that of the child's. This would imply that the child's dose should be a quarter of the adult's dose. More generally, the ratio of the child's to the adult's dose should be the square of the ratio of their heights. In general, if the standard height of an adult is, say, 1.7 m and a patient has a height of h m, then the patient's dose would be given by the formula:

$$\text{dose} = (h/1.7)^2 \times \text{standard adult dose}$$

This is all very well, except that in hospital it is much easier to measure the mass of very sick people than their heights (they may be too ill to stand up straight, but they can be weighed sitting or lying down). So what is really required is a formula which will allow someone to calculate a person's dosage from his or her mass. People are made out of the same sort of stuff—flesh and bone—however old or big they are; this means that mass is roughly proportional to volume, and volume in turn is proportional to the cube of the height.

So the formula works like this: if you know the dose for an average adult, follow these instructions to calculate the dose for anyone else, adult or child.

- (a) Weigh the patient.
- (b) Divide the patient's mass by the mass of the average adult (to which the standard dose is matched).
- (c) Take the cube root of the result (which gives the ratio of the patient's to the average adult's height).
- (d) Square the result (which gives the ratio of the surface area of the patient's kidneys to that of the average adult's).
- (e) Finally, multiply the dose for the average adult by this numerical factor.

These instructions are in the same form as a 'Think of a number' sequence, except that the 'number you first thought of' is the patient's mass. Here is how this works in detail, for a drug called phenobarbital, which is an anti-epileptic drug. The maintenance dose (the dose required to keep the concentration steady) of phenobarbital for an average adult is

In *Unit 4*, the formula for calculating the standard deviation followed a sequence of instructions like this.

100 milligrams per day; this is calculated on the assumption that the average adult weighs 70 kg. To calculate the maintenance dose, d milligrams per day, for a patient weighing w kg, carry out the steps described above in turn.

- (a) Weigh the patient:

$$w$$

- (b) Divide the patient's mass by the mass of the average adult:

$$\frac{w}{70}$$

- (c) Take the cube root of the result:

$$\sqrt[3]{\frac{w}{70}}$$

- (d) Square this result:

$$\left(\sqrt[3]{\frac{w}{70}}\right)^2$$

- (e) Finally, multiply the 100 milligram dose for the average adult by this numerical factor:

$$\left(\sqrt[3]{\frac{w}{70}}\right)^2 \times 100 \text{ milligrams}$$

That is to say, the dose in milligrams d is given by

$$d = 100 \left(\sqrt[3]{\frac{w}{70}}\right)^2$$

Activity 58 Daily dose

What is the daily dose of phenobarbital for a child whose mass is 35 kg? Use your calculator, or the fact that $\sqrt[3]{0.5}$ is about 0.8.

Activity 59 Theophylline

Theophylline is a drug used in the treatment of asthma; the maintenance dose of theophylline for an average adult is 200 milligrams every four hours. Find the formula which gives the maintenance dose of theophylline in terms of the patient's mass.

The formula for drug doses, on its own, is not of much value to a busy doctor in a hospital: just imagine the scene, with the ambulance bringing in a child with an asthma attack, fighting for breath, while the house officer (who may already have been on weekend duty for sixty hours

without sleep) tries to remember ‘Is it the cube root of the square or the square root of the cube?’ What is needed is a table showing the various values of the dose for conveniently spaced out values of the patient’s mass. Your calculator is equipped to provide just such a table.

Activity 60 *Drug-dose table*

Use your calculator to draw up a table of doses of theophylline for use in a hospital. You should decide what would be a sensible range of values of the patient’s mass, and adjust the table settings accordingly.

A graph of this function could also be useful for doctors to see how the dose varies with body mass. Use your calculator to produce a graph for people up to 100 kg in mass.

In this section, you used some formulas for areas and volumes (together with the tabulating and graphing facilities of the calculator), to analyse an important medical problem concerning drug dosage levels.

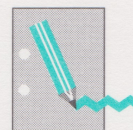
Outcomes

After studying this section, you should be able to:

- ◇ write down and use formulas for areas and volumes of simple figures (Activities 54 to 56);
- ◇ appreciate the consequences of the difference between the ways that areas and volumes scale when the linear dimensions of a solid figure are scaled (Activity 57);
- ◇ work with drug dose formulas to produce tables and graphs (Activities 58 to 60).

Unit outcomes

Activity 61 Looking back



Before you leave this unit, there are some things to complete. Go back to the Handbook activities on the advantages and disadvantages of algebra (Activity 12), and your compilation of meanings and rules (Activity 13), and make sure they are up to date. It is often useful to take time to think about what you have been doing and what you have learned before you rush off to the next piece of work. Just run through the unit in your mind's eye and make a mental list of what it covered and all the new vocabulary and techniques.

Look again at your Learning File activity sheet. How have you done on your self-assessment about algebra? If you have been using the outcomes to help you monitor your progress, which ones are you confident about and which (if any) do you still need to master? If you have found algebra difficult in the past, what do you feel about it now? Overall, what evidence can you provide to show what you have learned about the uses of symbols in mathematics? How could you demonstrate your progress? Think about which responses to activities you could use for this purpose.

Outcomes

After studying this unit, you should be able to:

- ◇ build up formulas from given information, including each step of a 'Think of a number' sequence;
- ◇ recognize whether or not a collection of symbols is an algebraic expression;
- ◇ describe in your own words the terms 'substitute', 'evaluate', 'value', and 'variable';
- ◇ evaluate an algebraic expression by substituting in a value for a variable;
- ◇ simplify algebraic expressions, multiply out brackets and multiply or divide an algebraic expression by a number;
- ◇ explain in your own words why a negative number times a negative number is a positive number;
- ◇ enter and store formulas in your calculator;
- ◇ use your calculator to evaluate a stored formula for particular values of the independent variable;

- ◇ use your calculator to produce tables and graphs for a formula;
- ◇ change the subject of a formula;
- ◇ solve a simple equation graphically or algebraically;
- ◇ use and interpret formulas, tables and graphs in a variety of contexts, such as converting units, travel planning, calculating areas and volumes, and adjusting drug dosages for mass;
- ◇ present notes or a log showing how your learning of algebra is developing;
- ◇ assess yourself based on criteria and monitor progress;
- ◇ use activities and feedback to help learning.

Comments on Activities

Activity 1

It is often useful to think about what you already know in relation to an area of work so that you can use it to ‘connect into’ the new work. Sometimes it is also helpful to think about how you feel on a particular area. Some students have strong feelings in relation to algebra, and in some cases, they may create a ‘block’. By thinking about and analysing what causes a difficulty or block it is often overcome.

Assessing yourself is an important part of becoming an effective learner. But you need something, that is, some criteria, to assess against. You may wish to use the unit outcomes or you may want to design criteria that are important to you. And do not forget to cite evidence that is valid and appropriate to help you assess as accurately as you can.

This ongoing activity will give you the opportunity to provide evidence for improving your learning.

You may wish to use your activity sheet as a log giving details of your progress. You could cite completed activities as evidence of your continuing progress. Remember to reference your log—perhaps giving dates or page numbers.

Activity 2

If a volume measures L litres and G gallons, then $L = 4.55 \times G$.

The litre equivalent of a pint is obtained by taking G to be $\frac{1}{8}$, and 1 pint is therefore $4.55 \times \frac{1}{8} = 0.57$ litres (to two decimal places).

Activity 3

If 1 mile is 1.61 km, then 1 km is $1 \div 1.61$ miles, which is approximately 0.62 miles. The required formula is therefore $M = 0.62 \times K$.

$$\begin{aligned} 40 \text{ km} &= 0.62 \times 40 \text{ miles} \\ &= 24.8 \text{ miles} \end{aligned}$$

To convert 24.8 miles to km, use $K = 1.61 \times M$. M is 24.8 so $K = 1.61 \times 24.8 = 39.93$ (to two decimal places), which is not quite 40 miles, due to rounding errors. (The reciprocal of 1.61 is not exactly 0.62.)

Activity 4

You must say something like: ‘multiply the number of cubits in Goliath’s height, namely 6, by 0.46, and add on a bit for the span (the span of your hand in metres)’. Now $0.46 \times 6 = 2.76$, while a span is about 23 cm, or 0.23 metres; so Goliath was about 3 metres tall.

Activity 5

Consider a temperature of $f^\circ\text{F}$. Subtracting 32 gives $f - 32$. Now multiply *the result* by $\frac{5}{9}$, to get:

$$c = \frac{5}{9} \times (f - 32)$$

A temperature of 212°F corresponds to $\frac{5}{9} \times (212 - 32) = \frac{5}{9} \times 180 = 100^\circ\text{C}$.

A temperature of 98.4°F corresponds to $\frac{5}{9} \times (98.4 - 32) = \frac{5}{9} \times 66.4 \simeq 36.9^\circ\text{C}$.

Activity 6

To change Celsius into Fahrenheit, you have to reverse the things done to change Fahrenheit to Celsius; that is, you have to do the reverse operations, *in the opposite order*. (This point will be discussed in greater detail in Subsection 4.1.)

Consider a temperature of $c^\circ\text{C}$. Multiplying by $\frac{9}{5}$ gives $\frac{9}{5} \times c$; adding 32 gives:

$$f = \frac{9}{5} \times c + 32$$

Activity 7

The time interval from 10 am to 2.15 pm is $4\frac{1}{4}$ or 4.25 hours. At this time, Alastair's distance from Cardiff is $67.5 + 45 \times 4.25 = 258.75$ miles, while Bethan's is $400 - 50 \times 4.25 = 187.5$ miles. So Alastair is further from Cardiff, and nearer to Edinburgh, than Bethan.

Activity 8

When the time is t hours, you have been travelling for $t - 10$ hours. When the trip meter reads d miles, you have travelled a distance of $d - 100$ miles. Your average speed s mph is given by:

$$s = \frac{d - 100}{t - 10}$$

At a quarter past eleven, the time on the twenty-four hour clock (in decimals, *not* hours and minutes) is 11.25. If the trip meter reads 180, the average speed since 10.00 is $(180 - 100) \div (11.25 - 10) = 80 \div 1.25 = 64$ mph.

Activity 9

- (a) Let the speed be r km/h and s mph. If you travel r kilometres in an hour, and 1 km is 0.62 miles, then you travel $0.62 \times r$ miles in an hour. Thus:

$$s = 0.62 \times r$$

- (b) A speed of 300 km/h is $0.62 \times 300 \simeq 186$ mph (some 32 mph in excess of the (1995) British train speed record).
- (c) Let the speed of s mph also be u metres per second. If you travel s miles in an hour, and 1 mile is 1.61 km, or 1610 metres, then you travel $1610 \times s$ metres in an hour. There are $60 \times 60 = 3600$ seconds in an hour, so if you travel $1610 \times s$ metres in an hour, you travel $(1610 \div 3600) \times s$ metres in a second; $1610 \div 3600 = 0.45$ (to two decimal places). Thus:

$$u = 0.45 \times s$$

A speed of 60 mph is $0.45 \times 60 = 27$ metres per second.

- (d) Let the time required to cover a mile be T seconds.

A speed of r km/h is $0.62 \times r$ mph; at this speed, you would take $1 \div (0.62 \times r)$ hours to cover a mile.

There are 3600 seconds in an hour, so this time is $3600 \div (0.62 \times r)$ seconds. Thus,

$$T = \frac{3600}{0.62 \times r} = \frac{5806}{r}$$

(since $3600 \div 0.62 \simeq 5806$).

- (i) It takes Eurostar $5806 \div 300 \simeq 19.3$, (just under 20) seconds to cover a mile.
- (ii) It takes the Earth $5806 \div 107\,000 \simeq 0.05$ seconds to cover a mile.
- (iii) It takes light $5806 \div 1\,100\,000\,000 \simeq 0.000\,005$ seconds to cover a mile.

Activity 10

- (a) The cost of hiring the car for five days at the daily rate, excluding the mileage, is $\pounds 38.75 \times 5 = \pounds 193.75$.
- (b) Suppose that Judy will travel m miles in the week. Then the additional cost, at $\pounds 0.19$ per mile, is $\pounds 0.19 \times m$. If the total cost is $\pounds p$, then:

$$p = 193.75 + 0.19 \times m$$

- (c) If Judy travels 150 miles, the cost is $\pounds 193.75 + 0.19 \times 150 = \pounds 222.25$, which is less than the weekly rate. For 250 miles, the cost is $\pounds 193.75 + 0.19 \times 250 = \pounds 241.25$ ($\pounds 19$ more), which is more than the weekly rate. Thus, the weekly rate would be the cheaper option.

Activity 11

- (a) Let the age of the hedge be A years, and let the number of different species be N . Then:

$$A = 110 \times N + 30$$

- (b) Twenty-eight species were counted in eight 30-yard sections, which is 3.5 species per 30 yards on average. The age of the hedge is therefore $110 \times 3.5 + 30 = 415$ years; it dates from just before 1600 according to Hooper's rule.

Activity 12

There are no comments on this activity.

Activity 13

There are no comments on this activity.

Activity 14

- (a) (i) When p takes the value 3, then $2 \times p - 2$ takes the value $2 \times 3 - 2$, which is 4.
 (ii) 12.33
 (iii) -4
 (b) When $M = 3$, then $3 \times M + 5 = 3 \times 3 + 5 = 14$.
 (c) When b is 5, then $2 + b - 2 \times b = 2 + 5 - 2 \times 5 = -3$.
 (d) When $x = 6$, then $1 + \frac{1}{3} \times x = 3$.

Activity 15

- (a) This does not make sense—there is a 'floating' minus sign at the end.
 (b) This makes sense.
 (c) This makes sense: the initial $+$ is fairly pointless, but not actually wrong (you might want to emphasize that the first number is $+4$ and not -4 , as in the following case).
 (d) This makes sense: the ' $-$ ' indicates a negative number, minus 4.

Activity 16

- (a) $3N$ (b) $\frac{2}{3}z$ (c) $56P^2$
 (d) $48x$

Activity 17

- (a) $24A$ (b) $21B$ (c) $10x$ (d) $6y$
 (e) $7p^2$ (f) $6A + 7A^2$ (This one does not simplify.)

Activity 18

- (a) $9x + x^2$ (b) $4x + 6x^2$
 (c) $6 + 6a + 3a^2$ (d) $14A + 2A^3$

(It does not matter if you have the terms in a different order: for example, $3a^2 + 6a + 6$ is the same as $6 + 6a + 3a^2$.)

Activity 19

When $N = 0.5$, then
 $4N + 10 = 4 \times 0.5 + 10 = 12$; while
 $N + 10 \times 4 = 0.5 + 40 = 40.5$.

Activity 20

- (a) When $N = 7$, then
 $4(N + 10) = 4 \times 17 = 68$
 $4N + 10 = 4 \times 7 + 10 = 38$
 $N + 10 \times 4 = 7 + 40 = 47$
 (b) $2(2N + 5)$; $\frac{1}{4}(4N + 8)$ or $(4N + 8) \div 4$.

Activity 21

- (a) (i) $9(m + 3)$ (ii) $(m + 3)(9 - 2m)$
 (b) (i) $5(m + 3)$ (ii) $3(4N + 10)$
 (iii) $5(9 - 2z)$

Activity 22

- (a) $2m + 6$ (b) $12X + 6$ (c) $45 - 10z$
 (d) $n^2 + 2n$ (e) $12X^2 + 6X$

Activity 23

$$(n + 2)(n + 3) = n(n + 3) + 2(n + 3) = n^2 + 3n + 2n + 6 = n^2 + 5n + 6$$

Activity 24

- (a) $y^2 + 8y + 7$
 (b) $p^2 + 4p + 4$
 (c) $p^2 + 4p + 4$ (Remember that $(p + 2)^2 = (p + 2)(p + 2)$.)
 (d) $2x^2 + 9x + 7$
 (e) $12p^2 + 14p + 4$

Activity 25

- (a) $m^2 + 8 + 15m$ is the odd one out (the others are all different versions of $(m + 3)(m + 5)$; the correct expression when the brackets are expanded is $m^2 + 8m + 15$).
 (b) $3x(x + 2) + 2(3x + 1)$ is the odd one out (the others are all different versions of $(3x + 1)(x + 2)$).

Activity 26

$x^2 + 2x + 6 + 3x$, $x^2 + 5x + 6$, and all the other possible orders of the same terms. But also: $x(x + 3) + 2(x + 3)$; $(x + 2)(x + 3)$; and therefore also $x(x + 2) + 3(x + 2)$.

Activity 27

- (a) $N + 2$ (b) $2m + 1$ (c) $5(2 - z)$ or $10 - 5z$

Activity 28

- (a) The result of following through the steps, starting with N , is

$$\sqrt{(N - 1)^2 + 2N - 1} - N. \text{ Now}$$

$$(N - 1)^2 = N^2 - 2N + 1, \text{ so}$$

$$(N - 1)^2 + 2N - 1 = N^2, \text{ and therefore}$$

$$\sqrt{(N - 1)^2 + 2N - 1} - N = \sqrt{N^2} - N = 0.$$

- (b) If you follow through the steps you get $2(3X + 10) - 8$. If you work this out, you get $6X + 12$. Division by 6 gives $X + 2$; take away X leaves 2. So the story could finish 'divide by 6; take away the number you first thought of; and your answer must be 2'.

- (c) There are many possibilities. Here is one: think of a number; quadruple it; add 10; halve the result; add 1; halve the result again; take away the number you first thought of; and the answer is 3.

Activity 29

$$P = \frac{J}{0.45}$$

Activity 30

It is because the bank intends to make some money on the deal. You would expect that, if you first change £1 into dollars, and then change the dollars back into pounds, you would finish up with less than you started with. So the factor in front of the S will be slightly less than $\frac{1}{4}$.

Activity 31

x is the mass of something in pounds (P in Activity 29); y is the mass of the same thing in kilograms (J); c is 0.45.

Activity 32

- (a) $x = y + 25$ (she says 'if I want to work out his age I must add twenty-five to mine').
 (b) Add 25 to undo the subtract 25 to get $y + 25 = x$, or $x = y + 25$.
 (c) This would have occurred if the story had been told first from my daughter's point of view, and her age had been called x and mine y . So it would have gone like this. She says 'my father's age is mine plus 25' or $y = x + 25$; I would have said 'my daughter's age is mine minus 25', or $x = y - 25$.

Activity 33

In order to make x the subject, you have to find that number which, when c is added to it, gives you y —and express your answer in terms of y . The number you require is $y - c$, so $x = y - c$.

Activity 34

$$c = k - 273$$

Activity 35

- (a) $x = \frac{y}{25}$ or $\frac{1}{25}y$
- (b) $X = Y - 7$
- (c) $q = 81p$
- (d) $x = y - 6$
- (e) $n = \frac{5}{2}m$
- (f) $v = u + \pi$ (Remember that π is just a number!)

Activity 36

- (a) $x = \frac{1}{2}(y - 1)$
- (b) $x = 2(y + 1)$
- (c) $x = \frac{1}{2}y - 1$

Activity 37

- (a) $k = \frac{5}{9}(f - 32) + 273$
- (b) $f = \frac{9}{5}(k - 273) + 32$

Activity 38

- (a) Let y be the cost in guineas, x the number of miles ridden. Then $y = 5 + x/21$.
- (b) The rearranged formula is $x = 21(y - 5)$. So when $y = 12$, then $x = 21 \times 7 = 147$, so the suspect travelled 147 miles.

Activity 39

- (a) $\frac{5}{8}x = 12$.
- (b) $x = \frac{8}{5} \times 12 = 19.2$, so the distance to my house is just over 19 km.
(Check: $\frac{5}{8} \times 19.2 = 12$.)

Activity 40

Write the equation to be solved as
 $x/21 + 5 = 120$. Rearranging this equation to

make x its subject involves two steps. First subtract the 5, to give $x/21 = 120 - 5 = 115$. Then multiply by the 21, to give $x = 115 \times 21 = 2415$. The solution is therefore 2415.

(Check: $2415/21 + 5 = 115 + 5 = 120$.)

Activity 41

- (a) $x = 2$
- (b) $x = 3$
- (c) $x = \frac{3}{7}$

Check your solutions by substituting your answers back into the original equation.

Activity 42

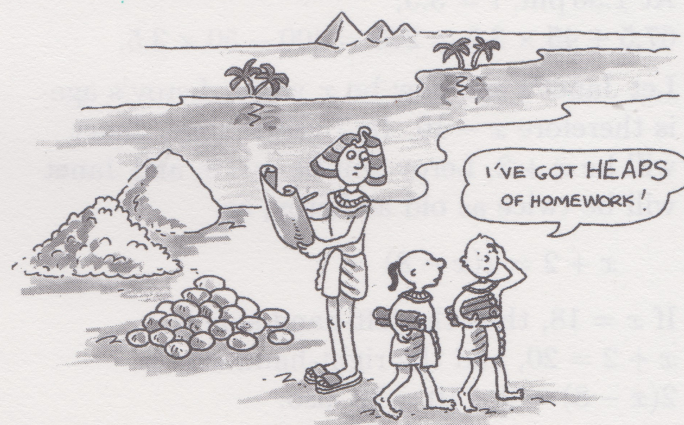
$3x + 7 = 1$; all the other equations are rearrangements of each other.

Activity 43

Let x stand for heap. Then 'heap and a seventh of heap' is $x + \frac{1}{7}x = \frac{8}{7}x$. This must equal 19, so the equation is:

$$\frac{8}{7}x = 19$$

The solution is $x = \frac{7}{8} \times 19 = \frac{133}{8}$. (Aren't you glad you are not studying in ancient Egypt?)



Activity 44

You need to solve the following equation:

$$-40 = \frac{5}{9}(f - 32)$$

The solution is given by:

$$\frac{9}{5}(-40) = \frac{9}{5} \times \frac{5}{9}(f - 32)$$

$$\frac{9}{5}(-40) = f - 32$$

Thus:

$$f = \frac{9}{5} \times (-40) + 32 = -72 + 32 = -40$$

Activity 45

When $x = -40$, then

$$\frac{5}{9}(x - 32) = \frac{5}{9} \times (-72) = -40 = x.$$

When $x = 40$, then

$$\frac{5}{9}(x - 32) = \frac{5}{9} \times (8) = -\frac{40}{9} \neq x,$$

so $x = 40$ is not a solution of this equation.

Note: $\neq$ means 'does not equal'.

Activity 46

The solutions are:

(a) $x = 2$

(b) $x = 2$

(c) $x = -4$

Activity 47

(a) The equation is $67.5 + 45t = 400 - 50t$.

At 1.30 pm, $t = 3.5$;

$$67.5 + 45 \times 3.5 = 225 = 400 - 50 \times 3.5.$$

- (b) Let Janet's age now be x years. Leroy's age is therefore $x - 10$. In 2 years' time, Janet will be $x + 2$, Leroy will be $x - 8$, and Janet will be twice as old as Leroy; so:

$$x + 2 = 2(x - 8)$$

If $x = 18$, then the left-hand side is $x + 2 = 20$, and the right-hand side is $2(x - 8) = 2 \times 10 = 20$ also.

- (c) Suppose there are x people in my house; so there are also x pets. There are therefore $2x$

heads, x tails, $2x + 4x$ feet; and so:

$$(2x)x = 2x + 4x$$

or

$$2x^2 = 6x$$

So $x = 3$. If $x = 3$, then $2x^2 = 2 \times 9 = 18$, while $6x = 6 \times 3 = 18$.

Activity 48

Let such a man's life expectancy be x years.

Then he spends $\frac{1}{15}x$ years as an infant, $\frac{2}{15}x$ years as a schoolboy, $\frac{1}{25}x$ years as a lover, $\frac{1}{5}x$ years as a soldier, $\frac{2}{5}x$ years as a justice, 10 years as a pantaloons, and 2 years in second childishness.

The total length of his life is obtained by summing these up; so they must together total x years. Thus

$$\frac{1}{15}x + \frac{2}{15}x + \frac{1}{25}x + \frac{1}{5}x + \frac{2}{5}x + 10 + 2 = x$$

or

$$\frac{63}{75}x + 12 = x$$

or

$$\frac{21}{25}x + 12 = x$$

Thus

$$\frac{4}{25}x = 12$$

so

$$x = \frac{25}{4} \times 12 = 75$$

The man's life expectancy is therefore seventy-five years.

Activity 50

Solutions are $x \simeq 0.38$ and $x \simeq 2.62$.

Activity 51

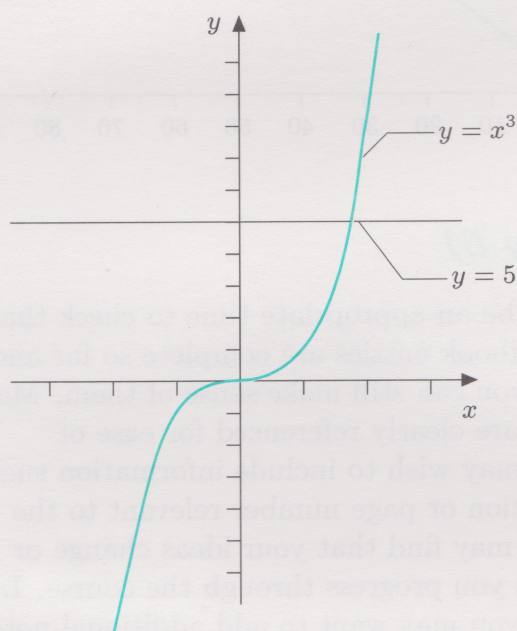
- (a) Someone who is touring will want to convert fairly large distances, perhaps several hundred miles; but will probably not require very high accuracy. The table difference could be set to 10. The graph could cover the range from $M = 100$ to $M = 500$, or so.

- (b) Distances on signposts for footpaths and cycle tracks are given to the nearest quarter of a mile; such distances are rarely more than 10 miles. The table difference should therefore be set at 0.25 or less. The graph should cover the range from $M = 0$ to $M = 12$, or so.

This activity is discussed on the associated audiotape.

Activity 52

The graph should look like the following.



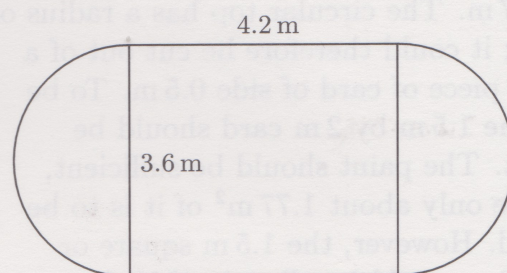
The value of $5^{1/3}$ (entered on the calculator as $5^{(1\div3)}$) is 1.709975947 (to nine decimal places).

Activity 53

The mileage at which becomes cheaper to take the weekly rate is just above 217 (to be overly accurate, 217.1052632). Above this, the weekly rate is cheaper; below this, the daily rate is cheaper. The cut-off occurs where the two are equal: that is, where $193.75 + 0.19m = 235$. The point of the question, therefore, was to solve this equation using your calculator by plotting the

graph of $P = 193.75 + 19m$ and finding where $P = 235$. You could also solve it algebraically.

Activity 54



The area of the rectangular piece is $4.2 \times 3.6 \text{ m}^2$. The two half circle pieces combine to form a circle of diameter 3.6 m, and area $\pi \times (1.8)^2 \text{ m}^2$. The total area is approximately 25.3 m^2 (so you would probably round it up to 26 m^2).

Activity 55

- (a) $V = \frac{4}{3}\pi r^3$
 (b) The rectangular part has sides l and $2\pi r$; there are two circular ends, each of radius r . The surface area A is given by:

$$A = 2\pi r l + 2\pi r^2$$

Activity 56

- (a) Using part of the formula from the previous activity, the curved surface area required is $2\pi \times 0.2 \times 0.8$, which is 1.00531 m^2 . This is the mathematical solution to the problem. The real-life solution is slightly different, due to the actual dimensions of the shape rather than its area.

The dimensions of the curved surface of the cylinder are 0.8 by $2\pi \times 0.2 \text{ m}$, which is 0.8 m by 1.26 m ; so the piece of card is not big enough.

- (b) The area this time is given by

$$A = 2\pi r l + \pi r^2$$

because the post box 'cylinder' has a top but no base. The area for the given dimensions is $2\pi \times 0.25 \times 1 + \pi \times 0.25^2$,

which is 1.767145868 m^2 . This is the mathematical solution, once again. What has been ignored?

The dimensions of the curved surface of the cylinder are 1 m by $2\pi \times 0.25 \text{ m}$ which is 1 m by 1.57 m . The circular top has a radius of 0.25 m ; it could therefore be cut out of a square piece of card of side 0.5 m . To be safe, the 1.5 m by 2 m card should be bought. The paint should be sufficient, because only about 1.77 m^2 of it is to be painted. However, the 1.5 m square or $1 \text{ m} \times 2 \text{ m}$ would be all right if Niels were happy to have a radius of just under 0.25 m .

Activity 57

This activity is discussed in the text following it.

Activity 58

About 63 or 64 milligrams.

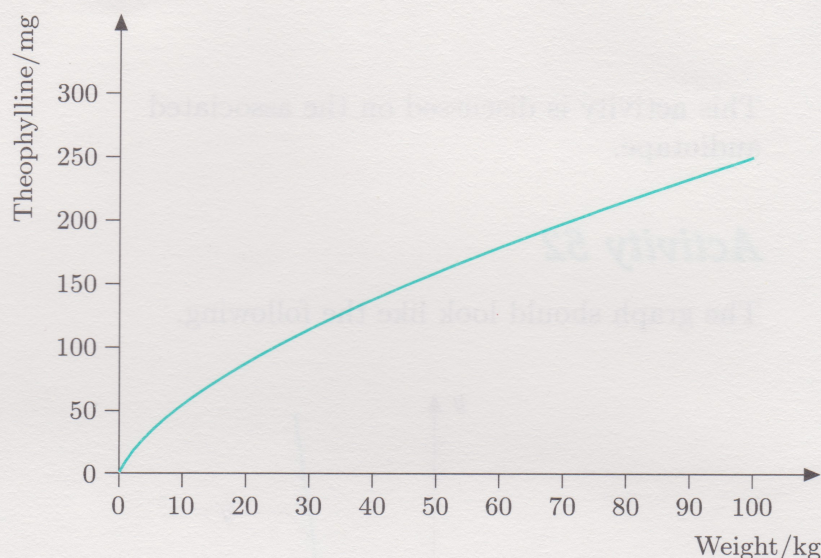
Activity 59

$$d = 200 \left(\sqrt[3]{\left(\frac{w}{70}\right)} \right)^2 \text{ every 4 hours.}$$

Activity 60

For the table, a table difference of 5 is probably adequate. It is doubtful, given all the minor differences between one person and another, that the drug dosage for an ordinary drug needs to be known any more accurately. (Of course,

drugs for special treatments may have to be calculated very carefully indeed.) The graph below has been drawn from about $w = 0$ to $w = 100$. (Heavy patients, and babies, probably have to be treated differently, so care needs to be taken using values at either end.) The graph looks like the following.



Activity 61

This may be an appropriate time to check that your Handbook entries are complete so far and also that you can still make sense of them. Make sure they are clearly referenced for ease of use—you may wish to include information such as the section or page number relevant to the unit. You may find that your ideas change or develop as you progress through the course. In this case, you may want to add additional notes to explanations and techniques or include different examples. Including a date provides useful reference points for your developing ideas.

Acknowledgements

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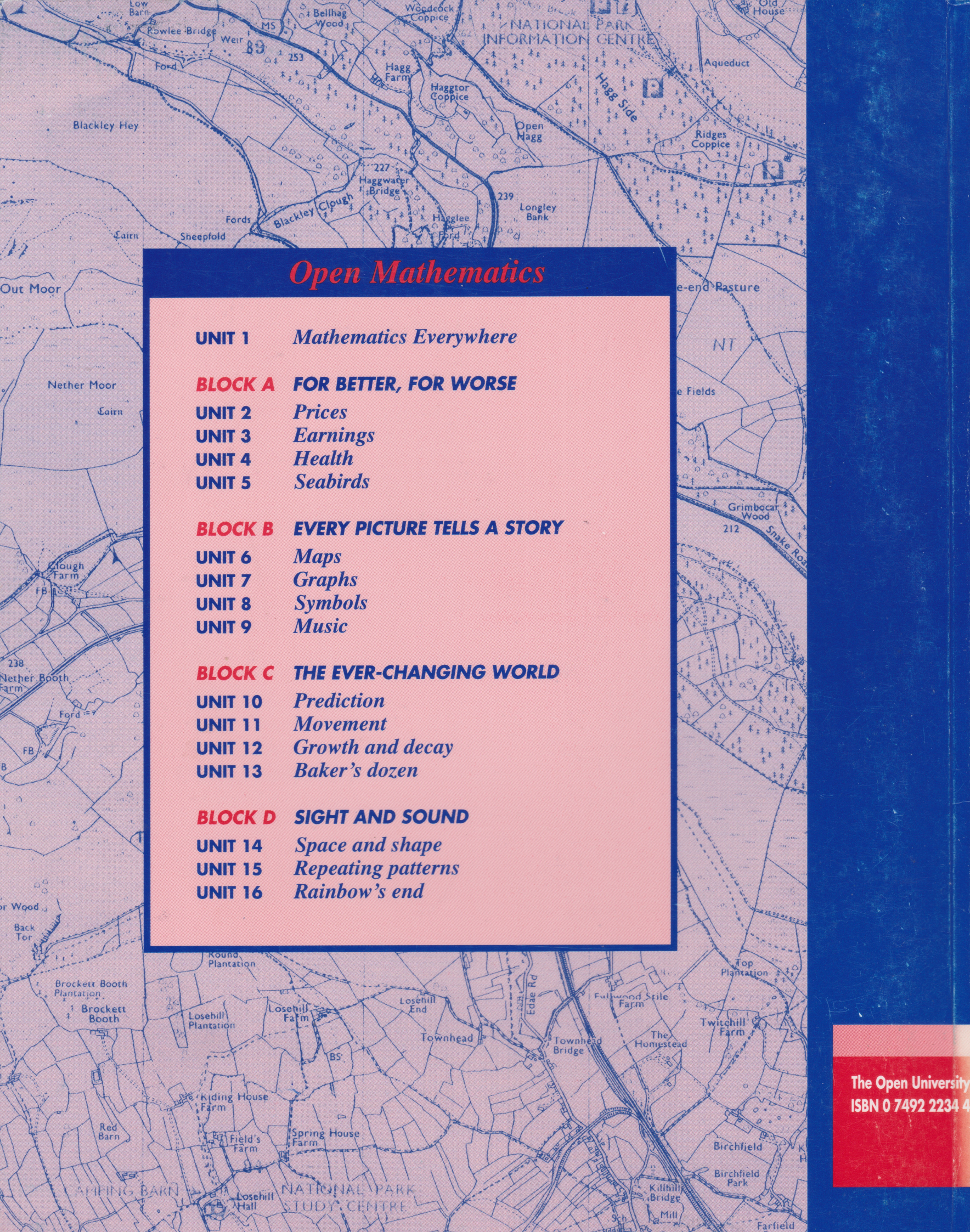
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